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DISTORTION OF BOUNDARY SETS UNDER INNER FUNCTIONS (II)

José L. Fernández, Domingo Pestana and José M. Rodríguez

We present a study of the metric transformation properties of inner functions of several complex variables. Along the way we obtain fractional dimensional ergodic properties of classical inner functions.

1. Introduction.

An inner function is a bounded holomorphic function from the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ into the unit disk Δ of the complex plane such that the radial boundary values have modulus 1 almost everywhere. If E is a non empty Borel subset of $\partial\Delta$, we denote by $f^{-1}(E)$ the following subset of the unit sphere $\mathbb{S}_n$ of $\mathbb{C}^n$

$$f^{-1}(E) = \left\{\xi \in \mathbb{S}_n : \lim_{r\to 1} f(r\xi) \text{ exist and belongs to } E\right\} .$$

The classical lemma of Löwner, see e.g. [**R**, p. 405], asserts that inner functions f, with $f(0) = 0$, are measure preserving transformations when viewed as mappings from $\mathbb{S}_n$ to $\partial\Delta$, i.e. if E is a Borel subset of $\partial\Delta$ then $|f^{-1}(E)| = |E|$, where in each case $|\cdot|$ means the corresponding normalized Lebesgue measure.

In this paper we extend this result to fractional dimensions as follows:

Theorem 1. *If f is inner in the unit disk Δ, $f(0) = 0$, and E is a Borel subset of $\partial\Delta$, we have:*

$$\operatorname{cap}_\alpha\left(f^{-1}(E)\right) \geq \operatorname{cap}_\alpha(E), \qquad 0 \leq \alpha < 1 .$$

Moreover, if E is any Borel subset of $\partial\Delta$ with $\operatorname{cap}_\alpha(E) > 0$, equality holds if and only if either f is a rotation or $\operatorname{cap}_\alpha(E) = \operatorname{cap}_\alpha(\partial\Delta)$.

Moreover, it is well known, see [**N**], that if f is not a rotation then f is ergodic, i.e., there are no nontrivial sets A, with $f^{-1}(A) = A$ except for a set of Lebesgue measure zero. This also has a fractional dimensional parallel.