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MAXIMAL SUBFIELDS OF $Q(i)$-DIVISION RINGS

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In this paper we determine the $Q(i)$-division rings which have maximal subfields of the form $E(i)$, where E/Q is cyclic and $i = \sqrt{-1}$. These are precisely the $Q(i)$-division rings having maximal subfields which are abelian over Q. More generally we determine the $Q(i)$-division rings having maximal subfields which are Galois over Q. We show that a division ring D contains such subfields if and only if the same is true for the 2-part of the Sylow decomposition of D.

1. Introduction and Preliminaries.

Let K be a field, and let D be a finite-dimensional K-central division ring. The dimension $[D : K] = m^2$ is a square, and one defines the index of D by $\operatorname{ind}(D) = m$. The maximal subfields of D are precisely those subfields which contain K and which have degree m over K. In case D has a maximal subfield L which is Galois over K, there exists a 2-cocycle $f : G \times G \to L^*$ such that D is isomorphic to the crossed product algebra $(L/K, f)$. This is proved in the chapter on simple algebras in [**Hers**], and we will assume familiarity with the results given there. It is well known that if K is a number field, then D has a maximal subfield which is cyclic of degree m over K. In [**Alb**], A.A. Albert posed the following rationality question: if F is a subfield of K, does there exist a cyclic extension E/F of degree m such that EK is a maximal subfield of D? He showed that such E need not exist, but considered conditions on $\operatorname{ind}(D)$ and $[K : F]$ under which such E could be found (e.g., Proposition 6, below).

The results of the present paper are motivated by this question in the special case $K = Q(i)$, $F = Q$. If E/Q is a cyclic extension of degree m such that $E(i)$ is a maximal subfield of a $Q(i)$-division ring D, then $E(i)$ is, in particular, an abelian extension of Q. It turns out that, conversely, if D has maximal subfields abelian over Q, then it has one of the form $E(i)$, where E/Q is cyclic. This raises the question of whether a $Q(i)$-division ring has maximal subfields which are cyclic, abelian, or even Galois over Q. We determine the $Q(i)$-division rings having such subfields in our main theorems 7, 8, and 12, according to the local indices of D. To define these,