A MEAN VALUE THEOREM FOR QUADRATIC FIELDS

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1. Introduction. Let K be an algebraic extension of the rationals of degree k, $F(n)$ denote the number of ideals whose norm is the rational integer n, $H(x)=\sum_{n\leq x}F(n)$. Let $\zeta(s, K)$ denote the Dedekind zeta function for the field K, that is,

$$\zeta(s, K)=\sum_{\mathfrak{A}}\frac{1}{N(\mathfrak{A})s}=\sum_{n=1}^{\infty}\frac{F(n)}{n^s}$$

and α the residue of $\zeta(s, K)$ at its simple pole at $s=1$.

It has long been known [8] that

$$H(x)=\alpha x+\Delta_k(x)$$

where

$$\Delta_k(x) = 0(x^{1-1/k})$$

and Landau [3] proved that

$$\Delta_k(x) = 0(x^{1-2/(k+1)})$$

The precise nature of the error term $\Delta_k(x)$ seems rather intractable and seems to be intimately related to the behavior of the function $\zeta(s, K)$ in the critical strip. Of considerable interest is the particular case when K is the Gaussian field $R(i)$, for in that case $\Delta_k(x)$ is the error term in the classical problem of the number of lattice points in a circle.

Using some results of class field theory, Suetuna [4] has obtained an improvement of Landau's result in the case when the field is normal and has abelian Galois group and $k\geqq 4$. For, when the field is abelian, the theorems of Weber-Takagi tell us that $\zeta(s, K)$ is the product of k Dirichlet L-functions belonging to primitive characters. Applying his approximate functional equation for the Dirichlet L-functions, and using refined estimates for these in the critical strip, Suetuna then obtains the desired result.

In the light of more recent techniques for dealing with the Riemann zeta function, further improvements are possible. The devices for handing the zeta function are used for the L-functions and the class

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