THE CALCULATION OF CONFORMAL PARAMETERS FOR SOME IMBEDDED RIEMANN SURFACES

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Introduction. Riemann surfaces were originally introduced as a tool for the study of multiple valued analytic functions. In Riemann's work they appear as covering surfaces of the complex plane with given branch points. Since then Riemann surfaces have been considered from several different aspects.

Here we shall follow the point of view assumed by Beltrami and Klein, who visualized these surfaces as two-dimensional submanifolds of Euclidean space whose conformal structure is defined by the surrounding metric.

Recent results of J. Nash[1] on isometric imbeddings of Riemannian manifolds assure that all models of Riemann surfaces with the natural Poincaré metric can be C^∞ isometrically imbedded in a sufficiently high (51) dimensional Euclidean space. However, the question still remains open whether or not every Riemann surface has a conformally equivalent representative in the ordinary three-dimmensional space.

Although the dimension requirement seems restrictive, there is reason to believe that, since only conformality is required, at least the compact surfaces can be conformally imbedded. We shall not be directly concerned here with this existence problem; instead, we shall present a family of elementary surfaces which may contain all conformal types and whose conformal structure can be easily characterized.

In the genus one case, the conformal structure is usually described by a complex parameter ν which gives the ratio of two principal periods of an abelian differential of the surface. It is always possible to choose these periods so that their ratio ν lies in the region $\mathfrak{M}$ of the Gauss plane defined by the inequalities:

$$\mathfrak{Im}\nu < 0,\ -\tfrac{1}{2} < \mathfrak{Re}\nu \leqq \tfrac{1}{2};\ |\nu| > 1 \text{ for } \mathfrak{Re}\nu < 0,\ |\nu| \geqq 1 \text{ for } \mathfrak{Re}\nu \geqq 0\,.$$

It is well known that every Riemann surface of genus one has in $\mathfrak{M}$ one and only one representative point.

It is easy to verify that the representative points ν of the tori of revolution lie in the imaginary axis and cover it completely. Thus it seems plausible that the affine images of the tori of revolution should cover all conformal types in the genus one case; however, we have

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[1] "The imbedding problem for Riemannian manifolds". *Annals of Mathematics*, 63 (1956), pp. 20-63.