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AN INTEGRAL INEQUALITY WITH APPLICATIONS TO THE DIRICHLET PROBLEM

JAMES CALVERT

An existence theorem for the elliptic equation $\Delta u - qu = f$ can be based on minimization of the Dirichlet integral $D(u, u) = \int |\nabla u|^2 + q\,|u|^2 dx$. The usual assumption that $q(x) \geqq 0$ is relaxed in this paper.

Actually the paper deals directly with the general second order formally self-adjoint elliptic differential equation $\sum_{i,k} D_i(a_{ik} D_k u) + qu = f$ where $q(x)$ is positive and "not too large" in a sense which will be made precise later. The technique consists in showing that the quadratic form whose Euler-Lagrange equation is the P.D.E. above is positive for a sufficiently large class of functions.

Earlier inequalities of Beesack [1] and Benson [2] show that there are positive functions $q(x)$ for which $\int |\nabla u|^2 - q\,|u|^2 dx \geqq 0$ for functions u which vanish on the boundary of the domain. D. C. Benson suggested to the author that this inequality might lead to existence theorems for $\Delta u + qu = f$.

Let $x = (x_1, x_2, \cdots x_n) \in R^n$. Let D be an open domain in R^n which may be unbounded unless the contrary is assumed. Let $C^\infty(D)$ denote the set of all infinitely differentiable complex-valued functions and $C_0^\infty(D)$ denote the subset of $C^\infty(D)$ of functions with compact support contained in D. Let $\|u\|_1^2 = \int_D \sum_{i=1}^n |D_i u|^2 + |u|^2 dx$ and let $C^{\infty *}(D)$ be the subset of $C^\infty(D)$ of functions with $\|u\|_1 < \infty$. Let $H_1(D)$ be the Sobolev space which is the completion of $C^{\infty *}(D)$ under $\|u\|_1$. For a function q of the special type encountered in §1, let $H_1^q(D)$ be the Sobolev space which is the completion of $C^{\infty *}(D)$ under the norm

$$\|u\|_q^2 = \int_D \sum_{i=1}^n |D_i u|^2 + q\,|u|^2 dx \;.$$

Let $\mathring{H}_1$ and $\mathring{H}_1^q$ be the completions of $C_0^\infty(D)$ with respect to $\|u\|_1$ and $\|u\|_q$. The reader who is not familiar with the Sobolev spaces can find a discussion of their calculus in Nirenberg [5].

1. An integral inequality.

THEOREM 1.1. *Let D be smooth enough to apply Gauss' Theorem. Let $a_{ik}(x)$ be hermitian positive definite, $a_{ik} \in C^1(D)$, and let $f_1, f_2, \cdots f_n$ be continuously differentiable complex valued functions of x, for all*