PACIFIC JOURNAL OF MATHEMATICS
Vol. 26, No. 1, 1968

AUTOMORPHISM GROUPS OF FINITE SUBGROUPS OF DIVISION RINGS

R. J. FAUDREE

If a finite group G can be embedded in the multiplicative group of a division ring, then G can be embedded in a division ring D generated by G such that any automorphism of G can be uniquely extended to be an automorphism of D. It seems natural then to investigate the relation between the automorphism group of G and the automorphism group of D.

We will prove that the automorphism group of G determines the automorphism group of D modulo the inner-automorphism group of D (i.e. every automorphism of D can be written as a product of an inner-automorphism of D and an automorphism of G). The automorphism group of G does not completely determine the automorphism group of D for the rational quaternions contain an isomorphic copy of Q_8, the quaternion group of order 8. There are infinitely many automorphisms of the rational quaternions but the automorphism group of Q_8 is finite.

Amitsur determined which finite groups can be embedded in a division ring [2]. We will use his conditions, but first some definitions will be given and certain algebraic structures will be discussed.

Let m and r be relatively prime integers, $s = (r-1, m)$ $t = m/s$ and $n =$ minimal integer satisfying $r^n \equiv 1 \pmod m$.

$$G_{m,r} = Gp(A, B \mid A^m = 1, BAB^{-1} = A^r, B^n = A^t) .$$

$\mathfrak{T}$, $\mathfrak{O}$, and $\mathfrak{J}$ will denote the binary tetrahedral, binary octahedral and binary icosahedral groups.

If ε_m is a primitive m^{th} root of unity and σ_r is the automorphism of $Q(\varepsilon_m)$ determined by the map $\varepsilon_m \rightarrow \varepsilon_m^r$, then

$$\mathfrak{A}_{m,r} = (Q(\varepsilon_m), \sigma_r, \varepsilon_m^t)$$

will denote the cyclic algebra determined by the field $Q(\varepsilon_m)$, the automorphism σ_r and the element ε_m^t. The map $A \rightarrow \varepsilon_m$ and $B \rightarrow \sigma_r$ determines an isomorphic embedding of $G_{m,r}$ into the algebra $\mathfrak{A}_{m,r}$. Under this identification we have

$$\mathfrak{A}_{m,r} = (Q(A), B, A^t) .$$

The algebra $\mathfrak{A}_{m,r}$ is a division algebra if and only if $G_{m,r}$ can be embedded in a division ring [2]. The following diagram gives some subalgebras of $\mathfrak{A}_{m,r}$ which will be of importance in this paper. Here $Z_{m,r}$ denotes the center of $\mathfrak{A}_{m,r}$.