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SIMPLE MODULES AND HEREDITARY RINGS

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The purpose of this note is to prove that if in a semi-primary ring Λ, every simple module that is not a projective Λ-module is an injective Λ-module, then Λ is a semi-primary hereditary ring with radical of square zero. In particular, if Λ is a commutative ring, then Λ is a finite direct sum of fields. If Λ is a commutative Noetherian ring then if every simple module that is not a projective module, is an injective module, then for every maximal ideal M in Λ we obtain $\mathrm{Ext}^1(\Lambda/M, \Lambda/M) = 0$. **The technique of localization now implies that** $\mathrm{gl.dim}\, \Lambda = 0$.

1. We say that Λ is a semi-primary ring if its Jacobson radical N is a nilpotent ideal, and $\Gamma = \Lambda/N$ is a semi-simple Artinian ring.

Throughout this note all modules (ideals) are presumed to be left modules (ideals) unless otherwise stated. For any idempotent e in Λ we denote by Ne the ideal $N \cap \Lambda e$.

We discuss first semi-primary rings Λ with radical N of square zero for which every simple module that is not a projective module is an injective module. We shall study the nonsemi-simple case, i.e., $N \neq 0$.

Under this assumption N becomes naturally a Γ-module.

Let e, e' be primitive idempotents in Λ for which $eNe' \neq 0$. In particular $Ne' \neq 0$. From the exact sequence $0 \to Ne' \to \Lambda e' \to S' \to 0$, it follows that S' is not a projective module since $\Lambda e'$ is indecomposable. Since S' is a simple module it follows that S' is an injective module.

Next consider the simple module $\Lambda e/Ne = S$. Since $eNe' \neq 0$, since Ne' is a Γ-module, and since on N the Γ-module structure and the Λ-module structure coincide, Ne' contains a direct summand isomorphic with S. This gives rise to an exact sequence $0 \to S \to \Lambda e' \to K \to 0$ with $K \neq 0$. If S were injective this sequence would split, and this contradicts the indecomposability of $\Lambda e'$. Therefore S is a projective module.

Hence Ne' is a direct sum of projective modules, therefore Ne' is a projective module. The exact sequence $0 \to Ne' \to \Lambda e' \to S' \to 0$ now implies $l.\mathrm{p.dim}\, S' \leqq 1$, and since S' is not a projective module, then $l.\mathrm{p.dim}\, S' = 1$.

Hence $l.\mathrm{p.dim}_\Lambda \Gamma = 1$, and this implies that Λ is an hereditary ring (i.e., $l.\mathrm{gl.dim}\, \Lambda = 1$) [1].

Conversely, assume that $l.\mathrm{gl.dim}\, \Lambda = 1$. Every ideal in Λ is the direct sum of $N_1, \cdots, N_t$ where N_1 is contained in the radical, and