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TRIVIALLY EXTENDING DECOMPOSITIONS OF E^n

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Let G be a monotone decomposition of E^n, then G can be extended in a trivial way, to the monotone decomposition G^1 of E^{n+1}, where $E^n = \{(x_1, \cdots, x_n, 0) \in E^{n+1}\}$, by adding to G all points of $E^{n+1} - E^n$. If the decomposition space E^n/G of G is homeomorphic to E^n, E^n/G is said to be obtained by a pseudo-isotopy if there exists a map $F: E^n \times I \rightarrow E^n \times I$, such that $F_t(=F \mid E^n \times t)$ is homeomorphism onto $E^n \times t$, for all $0 \leqq t < 1$, F_0 is the identity and F_1 is equivalent to the projection $E^n \rightarrow E^n/G$.

The purpose of this paper is to present a relation between these two notions. It will then follow, that if G is the decomposition of E^3 to points, circles and figure-eights, due to R. H. Bing, for which E^3/G is homeomorphic to E^3, then E^4/G^1 is not homeomorphic to E^4.

Moreover, we will present a direct, geometric proof to this particular property.

For definitions, see [1]. See also [2].

THEOREM 1. *If G is a monotone decomposition of E^n, such that E^n/G is homeomorphic to E^n, then the following are equivalent:*

(1) *E^{n+1}/G^1 is homeomorphic to E^{n+1}.*

(2) *E^n/G can be obtained by a pseudo-isotopy.*

Proof. (1) $\Rightarrow$ (2). Let $h: E^{n+1}/G^1 \rightarrow E^{n+1}$ be a homeomorphism and let $p: E^{n+1} \rightarrow E^{n+1}/G^1$ be the projection map.

The map $H: E^n \times I \rightarrow E^{n+1}$, defined by $H(x, t) = hp(x, 1 - t)$ for all $x \in E^n$, $t \in I$, is such that H_t is a homeomorphism into for all $0 \leqq t < 1$, H_1 is equivalent to the projection map $E^n \rightarrow E^n/G$, and $H(E^n \times I)$ is homeomorphic to $E^n \times I$, hence, up to a homeomorphism of $E^n \times I$ onto itself, H is the required pseudo-isotopy.

(2) $\Rightarrow$ (1). Let $F: E^n \times I \rightarrow E^n \times I$ be the pseudo-isotopy for E^n/G. The map $H: E^{n+1} \rightarrow E^{n+1}$, where

$$H(x, t) = \begin{cases} F(x, 1 + t) & -1 \leqq t \leqq 0 \\ F(x, 1 - t) & 0 \leqq t \leqq 1 \\ (x, t) & t \geqq 1 \text{ or } t \leqq -1 \end{cases} \qquad \text{where } x \in E^n .$$

is well defined, $H(E^{n+1}) = E^{n+1}$, and $H(E^{n+1})$ is homeomorphic to E^{n+1}/G^1, because $H_0 = F_1$ and it is equivalent to the projection map E^n onto E^n/G. The proof is completed.