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MEAN VALUE ITERATION OF NONEXPANSIVE MAPPINGS IN A BANACH SPACE

CURTIS L. OUTLAW

This paper applies a certain method of iteration, of the mean value type introduced by W. R. Mann, to obtain two theorems on the approximation of a fixed point of a mapping of a Banach space into itself which is nonexpansive (i.e., a mapping which satisfies $\|Tx - Ty\| \leqq \|x - y\|$ for each x and y).

The first theorem obtains convergence of the iterates to a fixed point of a nonexpansive mapping which maps a compact convex subset of a rotund Banach space into itself.

The second theorem obtains convergence to a fixed point provided that the Banach space is uniformly convex and the iterating transformation is nonexpansive, maps a closed bounded convex subset of the space into itself, and satisfies a certain restriction on the distance between any point and its image.

We note that a rotation T about zero of the closed unit disc in the complex plane satisfies the conditions of Theorems 1 and 2, but the usual sequence $\{T^n x\}$ of iterates of x does not converge unless x is zero.

DEFINITIONS. If Y is a Banach space, T is a mapping from Y into itself, and $x \in Y$, then $M(x, T)$ is the sequence $\{v_n\}$ defined by $v_1 = x$ and $v_{n+1} = (1/2)(v_n + Tv_n)$.

Following Wilansky [3, pp. 107–111], we say that a Banach space Y is *rotund* provided that if $w \in Y$, $y \in Y$, $w \neq y$, and $\|w\| = \|y\| \leqq 1$, then $(1/2)\|w + y\| < 1$.

THEOREM 1. *Let Y be a rotund Banach space, E be a compact convex subset of Y, and T be a nonexpansive mapping which maps E into itself. If $x \in E$ then $M(x, T)$ converges to a fixed point of T.*

Proof. If, for some n, $v_n = Tv_n$, then clearly $M(x, T)$ converges to v_n.

Hence suppose that $v_n \neq Tv_n$, for each n. Let z be a fixed point of T. Then $\{\|v_n - z\|\}$ is decreasing, for since Y is rotund and

$$\|Tv_n - z\| = \|Tv_n - Tz\| \leqq \|v_n - z\| ,$$

we have that

$$\|v_{n+1} - z\| = \left\|\frac{1}{2}(v_n + Tv_n) - z\right\| < \|v_n - z\| .$$