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SEMIGROUPS SATISFYING IDENTITY $xy = f(x, y)$

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Dedicated to Professor Keizo Asano on his Sixtieth Birthday

Let $f(x, y)$ be a word of length greater than 2 starting in y and ending in x. The purpose of this paper is to prove that a semigroup satisfies an identity $xy = f(x, y)$ if and only if it is an inflation of a semilattice of groups satisfying the same identity. As its consequence we find a necessary and sufficient condition for $xy = f(x, y)$ to imply commutativity.

Recently E. J. Tully has proved [7] that if a semigroup S satisfies an identity of the form $xy = y^m x^n$ then S is an inflation of a semilattice of abelian groups G_α's satisfying $x^k = 1$ for all $x \in G_\alpha$ where k is the greatest common divisor of $m - 1$ and $n - 1$; hence $xy = y^m x^n$ implies commutativity. This paper is to consider the general case of the right side of $xy = y^m x^n$ with the left side unchanged.

Let $f(x, y)$ denote a word involving both x and y, and $|f(x, y)|$ be the length of the word $f(x, y)$: $|x|_f$ be the number of x's which appear in $f(x, y)$; $|y|_f$ be also defined for y. For example if $f(x, y) = x^3y^2xy$, $|f(x, y)| = 7$, $|x|_f = 4$, $|y|_f = 3$. Throughout this paper we assume $|f(x, y)| > 2$, equivalently $|x|_f > 1$ or $|y|_f > 1$ or both.

Consider an identity of the form

$$xy = f(x, y) \tag{1}$$

in semigroups. A question is raised: Under what condition on $f(x, y)$ does (1) imply commutativity $xy = yx$? What we can say immediately is that $f(x, y)$ has to start in y. Because if $f(x, y)$ starts in x, then left zero semigroups of order > 1 satisfy (1) but are not commutative. For the similar reason $f(x, y)$ must end in x. From now on we assume $f(x, y)$ in (1) has the form:

$$\begin{cases} f(x, y) = y^{m_1}x^{n_1} \cdots y^{m_s}x^{n_s},\ m_i > 0,\ n_i > 0,\ i = 1, \cdots, s\,, \\ \text{and } |f(x, y)| > 2\,. \end{cases} \tag{2}$$

A semigroup D is called an inflation of a semigroup T if T is a subsemigroup of D and there is a mapping φ of D into T such that

$$\varphi(x) = x \qquad \text{for } x \in T$$

and

$$xy = \varphi(x)\varphi(y) \qquad \text{for } x, y \in D\,.$$

Let L be a semilattice. A semigroup S is called a semilattice L of semigroups S_α, $\alpha \in L$, if S is a disjoint union of $\{S_\alpha; \alpha \in L\}$ and