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COVERING SEMIGROUPS

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A topological semigroup is a Hausdorff space S together with a continuous associative multiplication $m: S \times S \rightarrow S$. The lifting of the group structure of a topological group to its simply connected covering space is a technique used in the theory of Lie groups. In this paper we investigate the lifting of the multiplication of a topological semigroup S to its simply connected covering space $(\bar{S}, \varphi)$. A general theory is developed and applications to examples are discussed.

1. Covering spaces. Let $\bar{S}$ and S be locally connected topological spaces and $\varphi: \bar{S} \rightarrow S$ a continuous map. If C is a subset of S, then C is *evenly covered* if $\varphi \mid \bar{C}: \bar{C} \rightarrow C$ is a homeomorphism for each component $\bar{C}$ of $\varphi^{-1}(C)$. If each point in S has an evenly covered open neighborhood, then φ is called a *covering map*. If φ is a covering map and $\bar{S}$ is connected, then $(\bar{S}, \varphi)$ is called a *covering space* of S. A covering space is called *trivial* if the covering map is a homeomorphism, and if S admits only trivial covering spaces, then S is called *simply connected*. If $(\bar{S}_1, \varphi_1)$ and $(\bar{S}_2, \varphi_2)$ are simply connected covering spaces of S and $\psi: \bar{S}_1 \rightarrow \bar{S}_2$ is a homeomorphism such that $\varphi_2 \circ \psi = \varphi_1$, then ψ is called a *covering space isomorphism*. An *automorphism* of $(\bar{S}, \varphi)$ is an isomorphism of $(\bar{S}, \varphi)$ with itself.

LEMMA 1. *Let $(\bar{S}, \varphi)$ be a covering space of S and T a connected space. If $\alpha, \beta: T \rightarrow \bar{S}$ are continuous maps with $\varphi \circ \alpha = \varphi \circ \beta$, then α and β agree everywhere or nowhere.*

LEMMA 2. *Let P be a topological space. Then P is simply connected if and only if* (a) *P is connected and locally connected and* (b) *if $\varphi: \bar{S} \rightarrow S$ is a covering map, $\psi: P \rightarrow S$ is continuous, p is in P, s is in $\bar{S}$ with $\psi(p) = \varphi(s)$, then there exists unique continuous $\bar{\psi}: P \rightarrow \bar{S}$ such that $\psi = \varphi \circ \bar{\psi}$ and $\bar{\psi}(p) = s$.*

LEMMA 3. *Let (P, ψ) and $(\bar{S}, \varphi)$ be covering spaces of S with p in P and s in $\bar{S}$ with $\psi(p) = \varphi(s)$. If P is simply connected and $\bar{\psi}: P \rightarrow \bar{S}$ is the unique lifting of ψ with $\bar{\psi}(p) = s$, then $\bar{\psi}$ is a covering map.*

LEMMA 4. *If $(\bar{S}_1, \varphi_1)$ and $(\bar{S}_2, \varphi_2)$ are simply connected covering spaces of S and s_i is in $\bar{S}_i$, $i = 1, 2$ with $\varphi_1(s_1) = \varphi_2(s_2)$, then there exists a unique covering space isomorphism $\psi: \bar{S}_1 \rightarrow \bar{S}_2$ such that $\psi(s_1) = s_2$.*