ON A THEOREM OF MURASUGI

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1. Let \( l = k_1 \cup k_2 \) be a 2-component link in \( S^3 \), with \( k_2 \) unknotted. The 2-fold cover of \( S^3 \) branched over \( k_2 \) is again \( S^3 \); let \( k_1^{(2)} \) be the inverse image of \( k_1 \), and suppose that \( k_1^{(2)} \) is connected. How are the signatures \( \sigma(k_1), \sigma(k_1^{(2)}) \) of the knots \( k_1 \) and \( k_1^{(2)} \) related? This question was considered (from a slightly different point of view) by Murasugi, who gave the following answer [Topology, 9 (1970), 283-298].

**Theorem 1 (Murasugi).**

Recall [4] that the invariant \( \xi(l) \) is defined by first orienting \( l \), giving, an oriented link \( l \), say, and then setting \( \xi(l) = \sigma(l) + \text{Lk}(k_1, k_2) \), where \( \sigma \) denotes signature and \( \text{Lk} \) linking number.

In the present note we shall give an alternative, more conceptual, proof of Theorem 1, and in fact obtain it as a special case of a considerably more general result.

The idea of our proof is the following. If \( l = l_1 \cup l_2 \) is a link, partitioned into two sublinks \( l_1 \) and \( l_2 \), then the 2-fold branched covers over \( l_1, l_2, \) and the whole of \( l \), are all quotients of a \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-cover branched over \( l \). After possibly multiplying by 2, the diagram consisting of these branched covers bounds a corresponding diagram of 4-manifolds, and the signatures of the various links involved are expressible in terms of the signatures of these 4-manifolds (and the euler numbers of the branch sets); see e.g., [3]. The result is then a consequence of a relation among these 4-manifold signatures (Lemma 1).

This more general setting requires that we consider links in 3-manifolds other than homology spheres; in §2 we discuss the signature in this context. (It becomes necessary to prescribe a particular 2-fold branched cover. However, we sacrifice some generality inasmuch as we restrict ourselves to oriented, null-homologous links: it would otherwise be necessary to prescribe a framing of the link as well.) In §3 we set up the diagram of covering spaces, and in §4 derive the relation between the signatures of the manifolds therein. Section 5 contains some consequences of this, including the appropriate generalization of Theorem 1.

All manifolds of dimensions 3 and 4 are to be oriented; manifolds of dimensions 1 and 2 are oriented only when this is explicitly