REPRESENTING POLYNOMIALS
BY POSITIVE LINEAR FUNCTIONS
ON COMPACT CONVEX POLYHEDRA

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If \( K \) is a compact polyhedron in Euclidean \( d \)-space, defined by linear inequalities, \( \beta_i \geq 0 \), and if \( f \) is a polynomial in \( d \) variables that is strictly positive on \( K \), then \( f \) can be expressed as a positive linear combination of products of members of \( \{\beta_i\} \). In proving this and subsidiary results, we construct an ordered ring that is a complete \( AGL(d, \mathbb{R}) \)-invariant for \( K \), and discuss some of its properties. For example, the ordered ring associated to \( K \) admits the Riesz interpolation property if and only if it is \( AGL(d, \mathbb{R}) \)-equivalent to a product of simplices. This is exploited to show that certain polynomials are not in the positive cone generated by the set \( \{\beta_i\} \).

Let \( L \) be a subfield of the real numbers, and let \( \beta_i = \sum_{1 \leq j \leq d} a_{ij} X_j + a_i, i + 1 \) \((i = 1, 2, 3, \ldots, s) \) be linear polynomials (“linear forms”) in the \( d \) variables \( \{X_j\} \), with coefficients from \( L \). Suppose the convex polyhedron in \( \mathbb{R}^d \) defined by \( K = \bigcap (\beta_i)^{-1}([0, \infty)) \) is compact and has interior. Let \( f \) be a polynomial in the \( d \) variables with entries from \( L \), such that the restriction, \( f \mid K \), is strictly positive. Then our first result (1.3) asserts that \( f \) may be represented as a combination with coefficients from \( L \cap \mathbb{R}^+ \) (that is, positive numbers in \( L \)) of terms that are products of the original set of \( \beta \)'s that determine \( K \). If \( f \) vanishes at only a vertex of \( K \) (and is strictly positive elsewhere), this decomposition does not hold in general (§III).

Our second principal result concerns the Riesz decomposition property in an ordered ring naturally associated to \( K \), and leads to some interesting geometric characterizations of those polytopes that are affinely homeomorphic to products of simplices. With \( K \) defined as above, define a monomial (in the \( \beta_i \)'s) to be a polynomial in the \( X \)'s that can be expressed as a product of the form

\[ \beta^w = \beta_1^{w(1)} \beta_2^{w(2)} \cdots \beta_s^{w(s)} \]

where \( w(k) \) is a non-negative integer, and \( w \) is the \( s \)-tuple \((w(1), w(2), \ldots, w(s))\). Define \( R_L[K] \) (or simply \( R[K] \) if there is no ambiguity about the coefficient field \( L \)) to be the polynomial ring,