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SOME COEFFICIENT PROBLEMS AND APPLICATIONS

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We combine Bombieri's method of second variation and Schiffer's variation by truncation to consider coefficient conjectures on $\log f'$, $\log f/z$, and $\log zf'/f$ where f is a normalized univalent analytic function in $|z| > 1$. Related results apply to the geometric properties of extremal functions for more general linear problems. Another by-product is a simplified approach to the geometric structure of solutions to the N th diameter problem.

1. Introduction. Let Σ be the class of one-to-one analytic functions $f(z) = z + \sum_{k=0}^{\infty} b_k z^{-k}$ in $\Delta = \{z : |z| > 1\}$. The function

$$K_n(z) = z(1 + z^{-n})^{2/n} = z + \frac{2}{n} z^{1-n} + \cdots$$

belongs to Σ and maps Δ onto the complement on an n-star γ_n. For $k = 2$ and $k = 3$ it is true that $|b_{k-1}| \le 2/k$ holds for the full class Σ, and for a time it was hoped that this estimate might persist for all k. However, Bazilevich [1] disproved this conjecture for even integers $k \ge 4$, and it is now known [3, 14] to be false for all integers $k \ge 4$.

In this article we shall consider similar conjectures for the coefficients of

$$\log f'(z) = \sum_{k=2}^{\infty} c_k z^{-k}, \quad \log \frac{f(z)}{z} = \sum_{k=1}^{\infty} d_k z^{-k}, \quad \text{and}$$

$$\log \frac{zf'(z)}{f(z)} = \sum_{k=1}^{\infty} e_k z^{-k}.$$

In the full class Σ the coefficients of d_k and e_k are unbounded. For that reason we shall consider these coefficients only for the compact subclasses Σ' consisting of functions in Σ that never vanish and Σ_0 consisting of functions in Σ with $b_0 = 0$. Since $d_1 = -e_1 = b_0$, the bounds $|d_1| = |e_1| = |b_0| \le 2$ are sharp for Σ', and these coefficients vanish for Σ_0. Thus we shall restrict attention from now on to coefficients with subscript $k \ge 2$.