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A Characterization of the Uniform Topology of a Uniform Space by the Lattice of its Uniformity.

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We shall denote in this paper by R a uniform space, and by $\{\mathfrak{M}_x \mid \mathfrak{X}\}$ its uniformity.[1] We denote by $\mathfrak{M}_x < \mathfrak{M}_y$ the fact that $\mathfrak{M}_x$ is a refinement of $\mathfrak{M}_y$ and by $\mathfrak{M}_x \vartriangle \mathfrak{M}_y$ the fact that $\mathfrak{M}_x^{\vartriangle} < \mathfrak{M}_y$. $\{\mathfrak{M}_x \mid \mathfrak{X}\}$ is a lattice by the order $<$, and has also the relation $\vartriangle$.

We shall show in this paper that in general a lattice-isomorphism between uniformities of two uniform spaces preserving the relations $\vartriangle$ and $<$ implies a uniform homeomorphism between the uniform spaces, and especially that when R has no isolated point, the structure of the lattice $\{\mathfrak{M}_x \mid \mathfrak{X}\}$ or of $\mathfrak{X}$ defines R up to a uniform homeomorphism.

An element of $\{\mathfrak{M}_x \mid \mathfrak{X}\}$, which is an open covering of R, is called simply a *u-covering* in this paper. German capitals are used for u-coverings but in 6 of the proof of Lemma 3.

Definition. Let $\mathfrak{M}$, $\mathfrak{N}$ be two u-coverings. We denote by $\mathfrak{M} \mathrel{\not\,\not<} \mathfrak{N}$ the fact that for every $M \in \mathfrak{M}$ there exists some $M' \in \mathfrak{M}$ such that $M \subset M'$ and $M' \not\subset N$ for all $N \in \mathfrak{N}$.

We denote by $\overline{\not\,\not<}$ the negation of $\not\,\not<$.

Lemma 1. *In order that* $\mathfrak{M} \mathrel{\not\,\not<} \mathfrak{N}$ *holds, it is necessary and sufficient that*

(1) $\mathfrak{M}^{\vartriangle}$ *contains no set consisting of one point,*

(2) *whenever* $\mathfrak{M} \not< \mathfrak{P}$, $\mathfrak{M} \not< \mathfrak{P} \smile \mathfrak{N}$ *holds.*[2]

Proof. Necessity: The condition (1) is obvious from the definition of $\not\,\not<$.

From $\mathfrak{M} \not< \mathfrak{P}$ we get $M \in \mathfrak{M}$ such that $M \not\subset P$ for all $P \in \mathfrak{P}$. Since $\mathfrak{M} \mathrel{\not\,\not<} \mathfrak{N}$, there exists $M' \in \mathfrak{M}$ such that $M' \supset M$, $M' \not\subset N$ for all $N \in \mathfrak{N}$.

1) Cf. J. W. Tukey, Convergence and uniformity in topology, (1940).
2) $\not<$ denotes the negation of $<$.