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Decomposition of radical elements of a commutative residuated lattice

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1. Recently E. Schenkman [4] has pointed out the similarlity between the properties of ideals in a commutative ring and of normal subgroups of a group. In particular he obtained that *every radical*[1] *A of a group G such that G/A has finite principal series has a unique minimal decomposition as an intersection of primes*[2].

In the present note we shall define a radicial element of a commutative residuated *cm*-lattice[3] L, and obtain a decomposition theorem for radical elements of L, which is a lattice-formulaltion of the above result and of the minimal decomposition theorem[4] of radical ideals in (commutative) Noetherian rings.

2. Let L be a commutative residuated *cm*-lattice with a greatest element e, and suppose that $ab \leq a$ for any two elements a and b of L[5].

For example, the lattice of all normal subgroups of any group forms a commutative residuated *cm*-lattice with above properties, if we define a multiplication $A \cdot B$ of normal subgroups A and B as the subgroup generated by all commutators $xyx^{-1}y^{-1}$ $(x \in A,\ y \in B)$[6].

For any element a of L, we define inductively $a^{(1)}=a$, $a^{(\rho)}=a^{(\rho-1)} \cdot a^{(\rho-1)}$ for $\rho>1$[7]. Then we have

(1) $a \leq b$ implies $a^{(\rho)} \leq b^{(\rho)}$,

(2) $\rho \leq \sigma$ implies $a^{(\rho)} \geq a^{(\sigma)}$,

(3) $(a \cap b)^{(\rho)} \leq a^{(\rho)} \cap b^{(\rho)}$,

(4) $(a \cdot a)^{(\rho)} = a^{(\rho)} \cdot a^{(\rho)}$,

(5) $a^{(\rho)(\sigma)} = a^{(\sigma)(\rho)}$,

(6) $a^{(\rho\sigma)} \leq a^{(\rho)(\sigma)}$,

(7) $(a \cup b)^{(\rho\sigma)} \leq a^{(\rho)} \cup b^{(\sigma)}$.

(1), …, (4) are immediate by induction on the whole number ρ.

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1), 2) Cf. [4, p. 376].

3) Cf. [1, p. 201]. The associative law for multiplication is not assumed.

4) Cf. [2, p. 202, Theorem 70].

5) The greatest element e is not necessarily a unity of L. If e is a unity then $ab \leq a$ for any two elements a and b of L.

6) Cf. [1, p. 204].

7) No confusion arises, even if we write $a^{\rho}=a^{(\rho)}$ for $\rho=1, 2$.