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Arithmetical ideal theory in semigroups

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The arithmetical ideal theory in rings may be regarded as that in semigroups, which can be treated as a generalization of the former. The arithmetical ideal theory in commutative semigroups was investigated for the firat time by Clifford ([4]) and then by Lorenzen ([8]). Some of the result of Clifford was extended to the noncommutative case by Kawada and Kondo ([7]). In the present paper we shall develop the arithmetical ideal theory in (noncommutative) semigroups, which is a generalization of that in noncommutative rings (cf. [1], [2] and [6]).

As preliminaries we deal in §1 with the factorization of integral elements in a lattice-ordered group and in §2 we give an abstract foundation of Artin-Hencke's ideal theory ([5]). Let S be a semigroup with unity quantity. The concepts of orders, maximal orders, ideals etc. in S are defined similarly as in rings. By using the results of §1, 2 we discus in §4 the theory of two-sided ideals with respect to a maximal order of S. We consider closed ideals (Lorenzen's r-ideals), i. e. ideals closed with respect to a given closure operation, by which a mapping of the set of all two-sided ideals in itself is defined. In order that the set of all closed two-sided $\mathfrak{o}$-ideals, $\mathfrak{o}$ a given regular order, forms an abelian group, which is a direct product of infinite cyclic groups, it is necessary and sufficient that Noether's axioms hold for $\mathfrak{o}$. Let $\mathfrak{o}$ be a regular order of S, for which Noether's axioms hold. The closure operation defined over two-sided $\mathfrak{o}$-ideals can be extended over $\mathfrak{o}$-sets containing regular elements. (A subset A of S is called a $\mathfrak{o}$-set if $\mathfrak{o}A = A\mathfrak{o} = A$.) A closed sub-semigroup of S containing $\mathfrak{o}$ is called a $\mathfrak{o}$-semigroup. We determine in §5 all $\mathfrak{o}$-semigroups. They form a Boolian algebra with respect to inclusion relation. In §7 we shall consider the Brandt's gruppoid of normal ideals. The factorization of integral normal ideals may be regarded as the factorization of integral elements in a lattice-ordered gruppoid, which will be treated in §6.

§1. Factorization of integral elements in a lattice-ordered group.

Let G be a lattice-ordered group (l-group) with unity quantity e. Elements of G will be denoted by small letters with or without suffices. We do not assume the multiplication to be commutative, except when we mention it particularly.