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Note on Chain Conditions in Free Groups

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§ 1. The purpose of this paper is to see what types of chain conditions hold in free groups.

In a previous paper[1] the author proved that the *maximum condition* holds on such subgroups of a free group F that are all generated by a finite number r of elements, for any prescribed natural number r. That is, if

$$H_1 \subseteqq H_2 \subseteqq \dots \quad \subseteqq H_n \subseteqq \dots$$

is a sequence of subgroups in a free group F and every H_n is generated by r elements, then the sequence is finite. This fact was proved also by H. Higman independently.[2]

Of course, in a free group the minimum condition on subgroups of the same type does not hold in general. But we can prove that a type of *restricted minimum condition* holds (Theorem 3). This can be proved as an immediate consequence of a theorem (Theorem 2) which is useful in some researches for sequences of subgroups in a free group.

The results obtained by M. Hall in his recent paper[3], on descending sequences of subgroups in a free group, are slightly generalized in ours (Theorem 4), and this generalization enables us to prove the result of F. Levi[4] too, from which the Hall's results can be derived immediately.

§ 2. Let F be a free group with a set $X=\{x_i\}$ of free generators. Any element f of F is uniquely expressible in its normal form $x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \dots x_{i_\lambda}^{\varepsilon_\lambda}$, where the x_i are elements out of X, $\varepsilon = \pm 1$, and two relations

$$x_{i_\nu} = x_{i_{\nu+1}} \quad \text{and} \quad \varepsilon_\nu + \varepsilon_{\nu+1} = 0$$

do not hold simultaneously for any $\nu = 1, 2, \dots, \lambda - 1$. The number $\lambda = \lambda(f)$ is called the *length* of f. The cardinal number of X is called the *rank* of F. The rank is determined uniquely by the group F, not depending on the choice of its set of free generators.

1) M. Takahasi: Note on locally free groups, Journ. of the Inst. of Polytechnics, Osaka City Univ. 1 (1950), 65-70.

2) H. Higman: A finitely related group with an isomorphic factor group, Journ. London Math. Soc. 26 (1951), 59-61.

3) M. Hall: A Topology for free groups and related groups, Ann. of Math. 52 (1950), 127-139.

4) F. Levi: Über die Untergruppen der freien Gruppen, Math. Zeitschr. 37 (1933), 90-97.