Shima, H. and Hao, J.-H.
Osaka J. Math.
37 (2000), 509–517

GEOMETRY ASSOCIATED WITH NORMAL DISTRIBUTIONS

HIROHIKO SHIMA and JIAN HUA HAO

(Received June 26, 1998)

1. Introduction

It is known that manifolds of smooth families of probability distributions admit *dualistic structures*. S. Amari proposed *Information Geometry*, whose keywords are *dualistic connections* [1]. Among them the case of dual connections being flat is interesting. Many important families of probability distributions, e.g. exponential families admit *flat* dual connections.

The notion of flat dual connections is the same with *Hessian structures* which have being developed from a different view point [9]-[12].

In this paper, for a linear mapping ρ of a domain Ω into the space of positive definite symmetric matrices we construct an exponential family of probability distributions $\{p(x;\theta,\omega)\}$ on $\mathbf{R}^n$ parametrized by $\theta \in \mathbf{R}^n$, $\omega \in \Omega$, and study a Hessian structure on $\mathbf{R}^n \times \Omega$ given by the exponential family. Such families contain n-dimensional normal distributions (Example 1) and a family of constant negative curvature (Example 2).

In case of a Lie group acting on Ω, ρ is assumed to be equivariant. O.S. Rothaus and I. Satake studied such a linear mapping ρ for homogeneous convex cones [7][8]. Using ρ we introduce a Hessian structure on a vector bundle over a compact hyperbolic affine manifold and prove a certain vanishing theorem (Theorem 2).

2. Hessian structures

We first review some fundamental facts on Hessian structures needed in this paper [1][11].

Let U be an n-dimensional real vector space with canonical flat connection D. Let Ω be a domain in U with a convex function ψ, i.e. the Hessian $Dd\psi$ is positive definte on Ω. Then the metric $g = Dd\psi$ is called a *Hessian metric* and the pair (D, g) a *Hessian structure* on Ω. Let $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ and $\{\mathbf{u}^{*1}, \cdots, \mathbf{u}^{*n}\}$ be dual basis of U and U^* (the dual vector space of U) respectively. We denote by $\{x^1, \cdots, x^n\}$ (resp. $\{x^*_1, \cdots, x^*_n\}$) the linear coordinate system with respect to $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ (resp. $\{\mathbf{u}^{*1}, \cdots, \mathbf{u}^{*n}\}$). Let $\iota : \Omega \longrightarrow U^*$ be a mapping given by

$$x^*_i \circ \iota = -\frac{\partial \psi}{\partial x^i},$$