Hoshino, M. and Sumioka, T.
Osaka J. Math.
36 (1999), 587–603

COLOCAL PAIRS IN PERFECT RINGS

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(Received September 12, 1997)

Our main aim of the present note is to provide several sufficient conditions for a colocal module L over a left or right perfect ring A to be injective. Also, by developing the previous works [8] and [5], we will extend recent results of Baba [1, Theorems 1 and 2] to left perfect rings and provide simple proofs of them.

Throughout this note, rings are associative rings with identity and modules are unitary modules. For a ring A we denote by $\operatorname{Mod} A$ (resp. $\operatorname{Mod} A^{\mathrm{op}}$) the category of left (resp. right) A-modules, where A^{op} denotes the opposite ring of A. Sometimes, we use the notation ${}_AL$ (resp. L_A) to signify that the module L considered is a left (resp. right) A-module. For a module L, we denote by $\operatorname{soc}(L)$ the socle, by $\operatorname{rad}(L)$ the Jacobson radical, by $E(L)$ an injective envelope and by $\ell(L)$ the composition length of L. For a subset X of a right module L_A and a subset M of A, we set $l_X(M) = \{x \in X | xM = 0\}$ and $r_M(X) = \{a \in M | Xa = 0\}$. Also, for a subset X of A and a subset M of a left module ${}_AL$ we set $l_X(M) = \{a \in X | aM = 0\}$ and $r_M(X) = \{x \in M | Xx = 0\}$. We abbreviate the ascending (resp. descending) chain condition as the ACC (resp. DCC).

Recall that a module L is called colocal if it has simple essential socle. We call a bimodule ${}_HU_R$ colocal if both ${}_HU$ and U_R are colocal. Let A be a semiperfect ring with Jacobson radical J. Let L_A be a colocal module with $H = \operatorname{End}_A(L_A)$ and $f \in A$ a local idempotent with $\operatorname{soc}(L_A) \cong fA/fJ$. In case L_A has finite Loewy length, we will show that L_A is injective if and only if ${}_HLf_{fAf}$ is a colocal bimodule and $M = r_{Af}(l_L(M))$ for every submodule M of Af_{fAf}. Also, in case A is left or right perfect and $\ell(Af/r_{Af}(L)_{fAf}) < \infty$, we will show that the following are equivalent: (1) L_A is injective; (2) ${}_HLf_{fAf}$ is a colocal bimodule and $r_{Af}(L) = 0$; and (3) ${}_HLf_{fAf}$ is a colocal bimodule and $M = r_{Af}(l_L(M))$ for every submodule M of Af_{fAf}.

Recall that a module L_A is called M-injective if for any submodule N of M_A every $\theta : N_A \to L_A$ can be extended to some $\phi : M_A \to L_A$. Dually, a module L_A is called M-projective if for any factor module N of M_A every $\theta : L_A \to N_A$ can be lifted to some $\phi : L_A \to M_A$. In case L is L-injective (resp. L-projective), L is called quasi-injective (resp. quasi-projective). Let A be a left perfect ring with Jacobson radical J and $e, f \in A$ local idempotents. Assume $\ell(Af/r_{Af}(eA)_{fAf}) < \infty$. Then we will show that eA_A is quasi-injective with $\operatorname{soc}(eA_A) \cong fA/fJ$ if and only if ${}_AE = E({}_AAe/Je)$ is quasi-projective with ${}_AE/JE \cong Af/Jf$ (cf. [1, Theorem 1]).