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DETERMINATION OF ALL QUATERNION CM-FIELDS WITH IDEAL CLASS GROUPS OF EXPONENT 2

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1. Introduction

In [10], we determined all normal octic CM-fields with class number one and noticed that class numbers of quaternion octic CM-fields are always even. Here, quaternion octic fields, or simply, quaternion fields are number fields whose Galois groups are isomorphic to the quaternion group of order 8. Later, the first author [9] determined all quaternion CM-fields with class number two: there is exactly one such number field. Now determination of CM-fields with ideal class groups of exponent ≤ 2 is a natural extension of class number one and two problems. In this paper, we prove:

Main Theorem *There are exactly two quaternion* CM*-fields with ideal class groups of exponent* 2*. Namely, the two following quaternion CM-fields:*

$$\boldsymbol{Q}\left(\sqrt{-(2+\sqrt{2})(3+\sqrt{6})}\right)$$

with discriminant $2^{24}3^6$ *and class number* 2, *and*

$$\boldsymbol{Q}\left(\sqrt{-\frac{5+\sqrt{5}}{2}\,\frac{5+\sqrt{21}}{2}\,(21+2\sqrt{105})}\right)$$

with discriminant $3^6 5^6 7^6$ *and class number* 8.

The proof consists of algebraic discussion, analytic discussion and numerical computation. In chapter 2, we determine possible forms of quartic subfields of quaternion CM-fields whose class groups have 4-rank being zero. Then, we determine possible forms, as radical extensions, of such quaternion CM-fields. In this determination, we use Fröhlich's description of quaternion fields [2], various results on quadratic fields (e.g. Rédei-Reichardt Theorem [11] and Scholz' Theorem [12]) and Kubota's description of bicyclic biquadratic fields [5]. In chapter 3, we give upper bounds on discriminants of quaternion CM-fields whose class groups have exponent 2 by using