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A SIX DIMENSIONAL COMPACT SYMPLECTIC SOLVMANIFOLD WITHOUT KÄHLER STRUCTURES

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1. Introduction

There are strong topological conditions for a compact manifold M of dimension $2n$ to admit a Kähler structure [20, 10]:

(i) the Betti numbers $b_{2i}(M)$ are non-zero for $1 \leq i \leq n$;
(ii) the Betti numbers $b_{2i-1}(M)$ are even;
(iii) $b_i(M) \geq b_{i-2}(M)$ for $1 \leq i \leq n$;
(iv) the Hard Lefschetz Theorem holds for M;
(v) the minimal model of M is formal (so in particular all Massey products of M vanish).

Gordon and Benson have proved that if a compact nilmanifold admits a Kähler structure then it is a torus [5]; more precisely they proved that the condition (iv) fails for any symplectic structure on a non-toral nilmanifold M. This result was independently proved by Hasegawa [12] by showing that (v) fails for M.

For a compact solvmanifold M of dimension 4 it is known that M has a Kähler structure if and only if it is a complex torus or a hyperelliptic surface. In fact, Auslander and Szczarba in [4] proved that if the first Betti number $b_1(M)$ of M is 2, M is a fiber bundle over T^2 with fiber T^2. Then by Ue [19] M has a complex structure only if it is a hyperelliptic surface or a primary Kodaira surface which is a compact nilmanifold. Thus, if M is a Kähler manifold, it must be a hyperelliptic surface. Since $1 \leq b_1(M) \leq 4$, M can be a Kähler manifold only if it is a complex torus or a hyperelliptic surface. The fact that a hyperelliptic surface is a solvmanifold follows from Auslander [3]. The above result may be generalized as the following conjecture: A compact solvmanifold has a Kähler structure if and only if it is a finite quotient of a complex torus.

In contrast to the case of compact nilmanifolds there are compact symplectic

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