Ishii, K. and Yamada, N.
Osaka J. Math.
30 (1993), 439–455

VISCOSITY SOLUTIONS OF NONLINEAR SECOND ORDER ELLIPTIC PDES INVOLVING NONLOCAL OPERATORS

Dedicated to Professor Hiroki Tanabe in commemoration of his sixtieth birthday

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(Received April 22, 1992)

1. Introduction

This paper deals with viscosity solutions of nonlinear degenerate elliptic partial differential equations (PDEs) involving nonlocal operators.

To begin with, we show model problems. Let $\Omega\subset \boldsymbol{R}^N$ be a bounded domain.

Model I. (*Integro-differential equation with obstacle*)

$$(1.1)\qquad \begin{cases} \max\{Lu-f, u-\varphi\} = 0 & \text{in } \Omega\,, \\ u(x) = \int_\Omega u(y)\, Q(dy, x) & \text{for } \ x\in\partial\Omega\,, \end{cases}$$

where L is an integro-differential operator of the form:

$$Lu(x) = -\sum_{i=1}^N g_i(x)\, u_{x_i}(x)+\alpha(x)\, u(x)+\lambda(x)\int_\Omega (u(x)-u(y))\, Q(dy, x)\,,$$

and $Q(\cdot, x)$ is a probability measure in Ω for $x\in\bar{\Omega}$.

Model II. (*Second order elliptic PDE with implicit obstacle*)

$$(1.2)\qquad \begin{cases} \max\{Lu-f, u-Mu\} = 0 & \text{in } \Omega\,, \\ \max\{u-g, u-Mu\} = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where L denotes the following linear (possibly degenerate) second order elliptic operator:

$$Lu(x) = -\sum_{i,j=1}^N a_{ij}(x)\, u_{x_i x_j}(x)+ \sum_{i=1}^N b_i(x)\, u_{x_i}(x)+c(x)\, u(x)$$

and Mu is a nonlocal term defined by

$$Mu(x) = \inf\, \{k(\xi)+u(x+\xi)\,|\,\xi\in(\boldsymbol{R}^+)^N,\ x+\xi\in\bar{\Omega}\}\ .$$

Model I is derived from the optimal stopping problem for piecewise-deterministic (PD) processes. S.M. Lenhart-Y.C. Liao [8] discussed the optimal