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GENERATORS IN GROTHENDIECK CATEGORIES WITH RIGHT PERFECT ENDOMORPHISM RINGS

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It is well-known that Grothendieck categories need not have projective objects (e.g. [12], 18.12). However, projective objects can be obtained from certain finiteness conditions. By a theorem of Năstăsescu, a Grothendieck category with an artinian generator has a finitely generated projective generator (see [10], [1]). His proof refers to the Gabriel-Popescu theorem as well as to the theory of Δ-injective modules. In this paper we give direct proofs of more general statements.

After preliminary results in section 1 we prove in section 2: Assume U is a generator in a Grothendieck category $\mathfrak{C}$ and the endomorphism ring of U is right perfect. Then there exists a projective generator in $\mathfrak{C}$. If the generator U is a coproduct of small objects U_λ in $\mathfrak{C}$ and $\hat{S}$, the ring of all endomorphisms of U with $(U_\lambda)f=0$ almost everywhere, is right perfect, then there exists a projective generator in $\mathfrak{C}$ which is a coproduct of small objects, and $\mathfrak{C}$ is equivalent to a full module category over a ring with enough idempotents.

In section 3 we obtain a new characterization of QF categories (in the sense of Harada [5]) by observing: In a locally finitely generated Grothendieck category $\mathfrak{C}$ an object U is a noetherian injective generator if and only if U is an artinian projective cogenerator.

A result of Auslander on categories of finite representation type ([3], Theorem 4.4) is generalized in section 4: A Grothendieck category $\mathfrak{C}$ of bounded representation type has a projective generator and is equivalent to a full module category over a ring with enough idempotents of bounded representation type.

Finally, in section 5, we interpret our results in a special case: For a left module M over an associative ring R denote by $\sigma[M]$ the full Grothendieck subcategory of R-Mod subgenerated by M. Assume $M=\oplus_\Lambda M_\lambda$, with finitely generated M_λ's, is a generator in $\sigma[M]$ and the ring $\hat{S}$ of all $f\in\mathrm{End}_R(M)$ with $(M_\lambda)f=0$ almost everywhere, is right perfect. Then there exists a projective left $\hat{S}$-module P which is a direct sum of local modules, such that $M\otimes_{\hat{S}}P$ is a projective generator in $\sigma[M]$, and $\sigma[M]$ is equivalent to $\widehat{\mathrm{End}}_{\hat{S}}(P)$-Mod. The observation on QF categories in section 3 yields new descriptions of quasi-projective noetherian QF modules in the sense of Hauger-Zimmermann [7],