M. Itoh
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THE MODULI SPACE OF YANG-MILLS CONNECTIONS OVER A KÄHLER SURFACE IS A COMPLEX MANIFOLD

MITSUHIRO ITOH

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1. Introduction

Let M be a compact, connected, oriented Riemannian 4-manifold. Let P be a smooth principal G-bundle over M. For simplicity we assume that the Lie group $G=SU(n)$, $n\geqq 2$. An $SU(n)$-connection A on P is called self-dual (anti-self-dual) if curvature form $F(A)=dA-A\wedge A$ satisfies $*F(A)=\pm F(A)$. Each self-dual (anti-self-dual) connection is characterized as a connection minimizing the Yang-Mills functional $\int_M |F|^2 dv$ and then gives a solution to the Yang-Mills equation. That the second Chern class $c_2(\mathfrak{g}^C)<0(>0)$ for the adjoint bundle $\mathfrak{g}$ of P is a topological restriction to P in order to admit a self-dual (anti-self-dual) connection. The moduli space $\mathcal{M}$ of self-dual (anti-self-dual) connections, namely, the orbit space of self-dual (anti-self-dual) connections with respect to the group $\mathcal{G}$ of gauge transformations has a structure of smooth manifold ([3], [7]).

A Kähler surface M with a Kähler metric g, which is certainly a Riemannian 4-manifold, carries the canonical orientation induced from the complex structure. Relative to this orientation a connection A is anti-self-dual if and only if its curvature is a 2-form of type (1,1) which is primitive (that is, orthogonal to the Kähler form ω). Therefore, by the integrability condition ([3]) each anti-self-dual connection induces a holomorphic structure on the complex adjoint bundle $\mathfrak{g}^C$. Since gauge-equivalent anti-self-dual connections give holomorphic structures which are isomorphic with respect to automorphisms of $\mathfrak{g}^C$, we have the canonical mapping from $\mathcal{M}$ to the moduli spcae of holomorphic structures on $\mathfrak{g}^C$. Furthermore an anti-self-dual $SU(n)$-connection A naturally defines an Einstein-Hermitian structure on the associated holomorphic vector bundle $\boldsymbol{E}=P\times_{SU(n)}\boldsymbol{C}^n$. We have also the fact that $\boldsymbol{E}$ is ω-semi-stable in the sense of Mumford and Takemoto ([9]). If A is moreover irreducible, then $\boldsymbol{E}$ is ω-stable. On the other hand, over a nonsingular projective surface the moduli space of holomorphic, rank two vector bundles of fixed Chern classes is a quasi-projective variety ([12]). From these reasons together with an easy observation that the moduli space $\mathcal{M}$