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A CHARACTERIZATION OF QF-ALGEBRAS

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We have defined a new class of rings in [3] which we call self mini-injective rings and we have noted that there exist artinian rings in the new class which are not quasi-Frobenius rings (briefly QF-rings).

We shall show in this note that if a ring R is an algebra over a field with finite dimension, then a self mini-injective algebra is a QF-algebra.

Throughout this note we assume a ring R contains the identity and every module is a unitary right R-module. We shall refer for the definitions of mini-injectives and the extending property, etc. to [3].

Let K be a field and R a K-algebra with finite dimension over K.

Theorem 1 (cf. [3], Theorems 13 and 14). *Let R be as above. Then the following conditions are equivalent.*

1) *R is self mini-injective as a right R-module.*

2) *R is self mini-injective as a left R-module.*

3) *Every projective right R-module has the extending property of direct decomposition of the socle.*

4) *Every projective left R-module has the extending property of direct decomposition of the socle.*

5) *R is a* QF-*algebra.*

Proof. R is self-injective as a left or right R-module if and only if R is a QF-algebra by [2]. In this case R is self-injective as both a right and left R-module by [1]. It is clear from [3], Theorem 3 and Proposition 8 that 1), 2) are equivalent to 3), 4), respectively. Hence, we may assume R is a basic algebra by [4] and [6].

1)$\rightarrow$5). Let $R=\sum_{i=1}^{n}\oplus e_iR$ be the standard decomposition, namely $\{e_i\}$ is a set of mutually orthogonal primitive idempotents and $e_iR\not\approx e_{i'}R$ if $i\neq i'$. Since R is right self mini-injective, R is right QF-2 by [3], Proposition 8 and $S(e_iR)\not\approx S(e_{i'}R)$ for $i\neq i'$ by [3], Theorem 5, where $S(\ )$ means the socle. Now e_iR is uniform as a right R-module and so the injective envelope $E(e_iR)$ of e_iR is indecomposable. We put $M^*=\mathrm{Hom}_K(M,K)$ for a K-module M. Then $E(e_iR)^*$