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ASYMPTOTIC SUFFICIENCY UP TO HIGHER ORDERS AND ITS APPLICATIONS TO STATISTICAL TESTS AND ESTIMATES

TAKERU SUZUKI

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1. Introduction. Suppose that n-dimensional random variable $z_n=(x_1, x_2, \cdots, x_n)$ is distributed according to a probability distribution $P_{\theta,n}$ parameterised by $\theta\in\Theta\subset R^1$, and each x_i is independently and identically distributed. In LeCam [1] it was shown that every estimator t_n with the form $t_n=\hat{\theta}_n+n^{-1}I^{-1}(\hat{\theta}_n)\cdot\Phi_n^{(1)}(z_n, \hat{\theta}_n)$ ($I(\theta)$ means Fisher information number), which is constructed using a reasonable estimator $\hat{\theta}_n$ and the logarithmic derivative $\Phi_n^{(1)}(z_n, \hat{\theta}_n)$ relative to θ of density of $P_{\theta,n}$, is asymptotically sufficient in the following sense; t_n is sufficient for a family $\{Q_{\theta,n};\ \theta\in\Theta\}$ of probability distributions and that

$$\lim_{n\to\infty}||P_{\theta,n}-Q_{\theta,n}||=0$$

uniformly on any compact set in Θ (where $||\cdot||$ means the totally variation of a measure). This implies that the statistic $(\hat{\theta}_n, \Phi_n^{(1)}(z_n, \hat{\theta}_n))$ is asymptotically sufficient up to order $o(1)$. As a refinement of this result it will be shown in this paper that for $k\geqq 1$ a statistic $t_n^*=(\hat{\theta}_n, \Phi_n^{(1)}(z_n, \hat{\theta}_n), \cdots, \Phi_n^{(k)}(z_n, \hat{\theta}_n))$, where $\Phi_n^{(i)}(z_n, \theta)$ means the $(i-1)$th derivative relative to θ of $\Phi_n^{(1)}(z_n, \theta)$, is asymptotically sufficient up to order $o(n^{-(k-1)/2})$ in the following sense; t_n^* is sufficient for a family $\{Q_{\theta,n};\ \theta\in\Theta\}$ and

$$\lim_{n\to\infty} n^{(k-1)/2}||P_{\theta,n}-Q_{\theta,n}||=o$$

uniformly on any compact subset of Θ. From our result it follows that if we use the maximum likelihood estimator $\hat{\theta}_n^*$ as the initial estimator $\hat{\theta}_n$ then the statistic $(\hat{\theta}_n^*, \Phi_n^{(2)}(z_n, \hat{\theta}_n^*), \cdots, \Phi_n^{(k)}(z_n, \hat{\theta}_n^*))$ is asymptotically sufficient up to order $o(n^{-(k-1)/2})$. In Ghosh and Subramanyam [4] it was mentioned that for exponential family of distributions $(\hat{\theta}_n^*, \Phi_n^{(2)}(z_n, \hat{\theta}_n^*), \Phi_n^{(3)}(z_n, \hat{\theta}_n^*), \Phi_n^{(4)}(z_n, \hat{\theta}_n^*))$ is asymptotically sufficient up to order $o(n^{-1})$ in pointwise sense relative to θ. Our result is more general and accurate one.

As an application of our result we try to improve arbitrarily given statistical tests or estimators. It will be shown that for arbitrarily given test sequence $\{\phi_n\}$, which is asymptotically similar of size α up to order $o(n^{-(k-1)/2})$ uniformly