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A GENERALIZATION OF MAGNUS' THEOREM

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Let $f(x, y)$ and $g(x, y)$ be polynomials in two variables with integral coefficients. O.H. Keller raised the problem in [1]: If the functional determinant $\partial(f, g)/\partial(x, y)$ is equal to 1, then is it possible to represent x and y as polynomials of f and g with integral coefficients? This problem drew many mathematicians' attension and several attempts have been made by enlarging the coefficient domain to the complex number field $\boldsymbol{C}$. But no success has been reported yet. On the other hand A. Magnus studied the volume preserving transformation of complex planes and obtained a result which is relevant to Keller's problem ([2]). From his results it is immediately deduced that Keller's problem is answered affirmativiely provided one of $f(x, y)$ and $g(x, y)$ has prime degree. For the proof Maguns used recursive formulas. But these formulas are complicated and not easy to handle. In this paper we shall give a simple proof of his theorem based on the notion of quasi-homogeneity for generalized polynomials. Moreover we shall go one step further than he did. Our results ensure that Keller's problem is valid provided one of $f(x, y)$ and $g(x,y)$ has degree 4 or larger degree is of the form $2p$ with an odd prime p. Since a complete solution of Keller's problem is not found yet our paper will be of some interest and worth-while publication.

1. Quasi-homogeneous generalized polynomials

Let x and y be two indeterminates. We shall set $\tilde{A}=\sum_{i,j\in\boldsymbol{Z}}\boldsymbol{C}x^iy^j$ where $\boldsymbol{C}$ is the complex number field and $\boldsymbol{Z}$ is the ring of rational integers. $\tilde{A}$ is a graded ring and the polynomial ring $\boldsymbol{C}[x, y]$ is a graded subring. Hereafter we shall call an element $f(x, y)$ of $\tilde{A}$ a generalized polynomial or simply a g-polynomial. We shall denote by $S(f)$ the set of lattice points (i, j) in the real two space $\boldsymbol{R}^2$ such that the monomial x^iy^j appears in $f(x, y)$ with a non-zero coefficient. $S(f)$ will be called the *supoprt* of $f(x, y)$. A g-polynomial $f(x, y)$ is called a homogeneous g-polynomial or a g-form if $S(f)$ lies in the straight line of the form $X+Y=m$ where $m\in\boldsymbol{Z}$ and is called the degree of the g-form $f(x, y)$.

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