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PROGRAM SIZE, ORACLES, AND THE JUMP OPERATION

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There are a number of questions regarding the size of programs for calculating natural numbers, sequences, sets, and functions, which are best answered by considering computations in which one is allowed to consult an oracle for the halting problem. Questions of this kind suggested by work of T. Kamae and D. W. Loveland are treated.

1. Computer programs, oracles, information measures, and codings

In this paper we use as much as possible Rogers' terminology and notation [1, pp. xv–xix]. Thus $N=\{0, 1, 2, \cdots\}$ is the set of (natural) numbers; $i, j, k, n, v, w, x, y, z$ are elements of N; A, B, X are subsets of N; f, g, h are functions from N into N; φ, ψ are partial functions from N into N; $\langle x_1, \cdots, x_k\rangle$ denotes the ordered k-tuple consisting of the numbers $x_1, \cdots, x_k$; the lambda notation $\lambda x[\cdots x\cdots]$ is used to denote the partial function of x whose value is $\cdots x\cdots$; and the *mu* notation $\mu x[\cdots x\cdots]$ is used to denote the least x such that $\cdots x\cdots$ is true.

The size of the number x, denoted $lg(x)$, is defined to be the number of bits in the xth binary string. The binary strings are: $\wedge$, 0, 1, 00, 01, 10, 11, 000,$\cdots$ Thus $lg(x)$ is the integer part of $\log_2(x+1)$. Note that there are 2^n numbers x of size n, and 2^n-1 numbers x of size less than n.

We are interested in the size of programs for a certain class of computers. The zth computer in this class is defined in terms of $\varphi_z^{(2)X}$ [1, pp. 128–134], which is the two-variable partial X-recursive function with Gödel number z. These computers use an oracle for deciding membership in the set X, and the zth computer produces the output $\varphi_z^{(2)X}(x, y)$ when given the program x and the data y. Thus the output depends on the set X as well as the numbers x and y.

We now choose the standard universal computer U that can simulate any other computer. U is defined as follows:

$$U^X((2x+1)2^z-1, y) = \varphi_z^{(2)X}(x, y)\,.$$

Thus for each computer C there is a constant c such that any program of size