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LIE ALGEBRAS OF DIFFERENTIAL OPERATORS

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The purpose of the present paper is to study the algebraic structure of the Lie algebra $\mathcal{D}(M)$ that consists of all the differential operators on a smooth manifold M. $\mathcal{D}(M)$ contains the Lie algebra $\mathcal{A}(M)$ of the vector fields on M as a subalgebra, which has been studied by many authors from various standpoints. Our investigation is motivated by Gelfand-Fuks's paper [1] concerning the cohomology theory of $\mathcal{A}(M)$. Indeed, their strong algebraic tendency has led us to expect that it will be fruitful to study differential operators from the viewpoint of Lie algebra.

Our main idea lies in regarding $\mathcal{D}(M)$ as a representation space of $\mathcal{A}(M)$ through the adjoint operations. This idea applies to the following two points. The first is to establish a kind of reducibility theorem with respect to this representation, which reveals certain characteristic features of the algebraic structure of $\mathcal{D}(M)$. The second is to consider the one-dimensional coholomogy group of $\mathcal{A}(M)$ associated with the representation, which yields a sufficient knowledge of the derivation space and the automorphism group of $\mathcal{D}(M)$.

We shall describe the outline of the present paper. Section 1 deals with basic notions and certain useful lemmas. Section 2 deals with $\mathcal{A}(M)$ and refers to Pursell-Shanks [5]. We give a characterization of the subalgebra that consists of the vector fields vanishing at a point of M. Using this characterization we can show that the algebraic structure of $\mathcal{A}(M)$ uniquely determines the smooth structure of M.

A subspace of $\mathcal{D}(M)$ is called an $\mathcal{A}$-space if it is invariant under the adjoint operations of $\mathcal{A}(M)$. For example, the space $\mathcal{D}_k(M)$ consisting of all the k-th order differential operators is an $\mathcal{A}$-space. In section 3, we give a structural theorem for $\mathcal{A}$-spaces, which states that any $\mathcal{A}$-space contained in $\mathcal{D}_k(M)$ coincides with one of a finite number of canonical $\mathcal{A}$-spaces in a neighborhood of all the points of M except those which lie in a nowhere-dense subset of M. From this structural theorem we can immediately deduce the theorems concerning ideals and those subalgebras which contain $\mathcal{A}(M)$, as we show in section 4.

In section 5, we determine the derivations of $\mathcal{D}(M)$ and certain subalgebras.

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