Nakao, S.
Osaka J. Math.
9 (1972), 513-518

ON THE PATHWISE UNIQUENESS OF SOLUTIONS OF ONE-DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

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(Received December 13, 1971)

Introduction

In this paper, we shall discuss a problem of the pathwise uniqueness for solutions of one-dimensional stochastic differential equations. Let $a(x)$ and $b(x)$ be bounded Borel measurable functions defined on R. We shall consider the following one-dimensional Itô's stochastic differential equation;

$$(1) \qquad dx_t = a(x_t)\,dB_t + b(x_t)\,dt.$$

K. Itô [1] proved that, if $a(x)$ and $b(x)$ are Lipschitz continuous, a solution is unique and it can be constructed on a given Brownian motion B_t. On the other hand, if $|a(x)|$ is bounded from below by a positive constant (i.e. uniformly positive), then a solution of (1) exists and it is unique in the law sense. This follows easily from a general result of one-dimensional diffusions (cf. [2]). However, though the distribution of $\{x., B.\}$ is unique, x_t is not always expressed as a measurable function of x_0 and $\{B_s, s\leqq t\}$. For example, if $a(x)=\text{sgn}\, x$, $a(0)=1$ and $x_0\equiv 0$, it is not difficult to see that $\sigma\{|x_s|\,;\, s\leqq t\}=\sigma\{B_s;\, s\leqq t\}$.

Here, we will show that, if $a(x)$ is uniformly positive and of bounded variation on any compact interval, then the pathwise uniqueness holds for (1). This implies, in particular, that x_t is expressed as a measurable function of x_0 and $\{B_s, s\leqq t\}$ (cf. [5]). In this direction, M. Motoo (unpublished) already proved that the pathwise uniqueness holds for (1) if $a(x)$ is uniformly positive and Lipschitz continuous and if $b(x)$ is bounded measurable. Also, T. Yamada and S. Watanabe [5] proved the pathwise uniqueness of (1) if $a(x)$ is Hölder continuous of exponent $\frac{1}{2}$ and $b(x)$ is Lipschitz continuous. Our above mentioned result may be interesting in a point that it applies for many discontinuous $a(x)$. It is still an open question whether only the uniform positivity of $a(x)$ implies the pathwise uniqueness.

Finally the author wishes to express his sincere gratitude to Professor S. Watanabe for his invaluable suggestions.