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ON REPRESENTATIONS OF DIRECT PRODUCTS OF FINITE SOLVABLE GROUPS

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Let K be a field and π a finite group. We denote by $G_0(K\pi)$ the Grothendieck ring of $K\pi$. Let π_i be a finite group and M_i be finitely generated $K\pi_i$-module, $i=1, 2$. Let us denote by $M_1 \# M_2$ the outer tensor product of M_1 and M_2. We can define the natural ring homomorphism φ: $G_0(K\pi_1)\otimes G_0(K\pi_2) \rightarrow G_0(K(\pi_1\times\pi_2))$ by putting $\varphi([M_1]\otimes[M_2])=[M_1\# M_2]$. In this paper we study the kernel and cokernel of φ.

1. Let π be a finite group, E a finite normal separable extension of K which is a splitting field of π, and $\mathcal{G}(E/K)$ the Galois group of E over K. Let N be an $E\pi$-module with character χ and $\sigma\in\mathcal{G}(E/K)$. Then we define an $E\pi$-module σN, the conjugate of N, as usual and denote it's character by $\sigma\chi$. We denote the Schur index of N over K by $m_K(N)$.

Now, let π be the direct product of finite groups π_1 and π_2, $\pi=\pi_1\times\pi_2$. Let M_i be an irreducible $K\pi_i$-module, $i=1, 2$, and denote an irreducible $E\pi_i$-component of $M_i^E=M_i\otimes_K E$ by N_i, the character of N_i by ψ_i and the Galois group E over $K(\psi_i)$ by $\mathcal{H}_i=\mathcal{G}(E/K(\psi_i))$. Then, the following results can be found in [3].

(1) If σ, $\tau\in\mathcal{G}(E/K)$, then $\sigma N_1\#\tau N_2$ is an irreducible $E[\pi_1\times\pi_2]$-module also and $m_K(N_1\# N_2)=m_K(\sigma N_1\#\tau N_2)$.

(2) $M_1\# M_2$ is completely reducible. $M_1\# M_2=k(T_1\oplus\cdots\oplus T_r)$, where the $\{T_i\}$ are nonisomorphic irreducible $K\pi$-modules and $k=m_K(N_1)m_K(N_2)/m_K(N_1\# N_2)$. The $\{T_i\}$ have common K-dimension s, where $s=m_K(N_1\# N_2)(K(\psi_1, \psi_2): K)(N_1\# N_2: E)$.

(3) $M_1\# M_2$ is an irreducible $K\pi$-module if and only if the following conditions are satisfied:

(a) $m_K(N_1)m_K(N_2)=m_K(N_1\# N_2)$.

(b) $\mathcal{G}(E/K)=\mathcal{H}_1\mathcal{H}_2$.

(c) $(K(\psi_1): K)(K(\psi_2): K)=(K(\psi_1, \psi_2): K)$.

(4) Let $\pi_1=\pi_2$, $\pi=\pi_1\times\pi_1$. Let M_1 be an irreducible $K\pi_1$-module. Then $M_1\# M_1$ is irreducible if and only if M_1 is an absolutely irreducible $K\pi_1$-module.

Since for any irreducible $K[\pi_1\times\pi_2]$-module M we can find a unique irreducible $K\pi_i$-module M_i, $i=1, 2$, satisfying $M_1\# M_2\oplus > M$, the following is an immediate corollary to (3).