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ON THE ANNIHILATOR IDEALS OF THE RADICAL OF A GROUP ALGEBRA

Dedicated to Professor K. Asano for his 60th birthday

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1. Introduction

Let G be a finite group, and k a field of characteristic p. Let $\mathfrak{n}$ denote the Jacobson radical of the group algebra kG, and $r(\mathfrak{n})$ the right annihilator ideal of $\mathfrak{n}$. In this paper we shall show some connections between $r(\mathfrak{n})$ and p-elements of G. One of them will state that $r(\mathfrak{n})$ contains the sum of all p-elements of G (including the identity). This may be regarded in a sense as a refinement of Maschke's theorem. In fact, if p does not divide the order of G then the identity is the only p-element, which implies $r(\mathfrak{n})\ni 1$ and hence $\mathfrak{n}=0$. On the other hand, as is easily seen from a theorem of T. Nakayama on Frobenius algebras (see §2), $r(\mathfrak{n})$ is a principal ideal. We shall show that it is generated by an element which is left invariant by every automorphism of kG induced by that of G. As an application of this fact, we shall give a lower bound for the first Cartan invariant in terms of the chief composition factors of G. The present study owes heavily to some general results on Frobenius algebras and symmetric algebras, which will be summarized in the next section.

NOTATION. If A is a ring, rad(A) will denote the Jacobson radical of A. For a subset T of A, $r(T)$ and $l(T)$ will denote respectively the set of right annihilators and the set of left annihilators of T in A. If M is a subset of a finite group G, then $\Delta_M=\sum_{\sigma\in M}\sigma\in kG$.

2. Preliminary results

Let $A(\ni 1)$ be a finite dimensional algebra over a field k.

DEFINITION. A linear function λ ($\in A^*=\mathrm{Hom}_k(A, k)$) is called *non-singular* if its kernel contains no left or right ideals other than zero. While, λ is called *symmetric if* $\lambda(ab)=\lambda(ba)$ for all a, $b\in A$.

If λ is a linear function and $a\in A$, we denote by λ_a the linear function defined