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A NOTE ON AZUMAYA'S THEOREM

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We say that a left Λ-module M is a cogenerator of the category of left Λ-modules if for every submodule N_1 of a left Λ-module N there exists a Λ-homomorphism f from N to M such that $f(N_1)\neq 0$. Let $\{I_\alpha\}$ be the full set of non isomorphic irreducible Λ-modules, and $\{E_\alpha\}$ be the set of their injective hulls. Then a left Λ-module M is a cogenerator if and only if M contains every E_α. In this case the sum of all E_α's is a direct sum by Zorn's Lemma (see Lemma 1[3]). A cogenerator is a faithful module (Lemma 2). The aim of this paper is to compare the ring for which every faithful module is a cogenerator with the ring for which every faithful module is generator (see G. Azumaya [1]). We assume every ring has units and every module is unitary.

Lemma 1. *Let M be a left Λ-module and A be an arbitrary set of index. Then the followings are equivalent.*

(1) *M is a cogenerator*

(2) *$\sum_{\nu\in A}^{\oplus} M_\nu$ $(M_\nu\cong M)$ is a cogenerator*

(3) *$\Pi_{\nu\in A} M_\nu$ $(M_\nu\cong M)$ is a cogenerator*

Proof. (1)$\Rightarrow$(2)$\Rightarrow$(3) is clear by Lemma 1 [3]. So we shall prove (3)$\Rightarrow$(1). Choose any E_α, then we have Λ-maps $E_\alpha \xrightarrow{\tau} \Pi M_\nu \xrightarrow{\pi_\nu} M_\nu$ where τ is a monomorphism and π_ν are the canonical maps. Let $f_\nu=\pi_\nu\cdot\tau$ then we see $\bigcap \ker f_\nu=0$. If $\ker f_\nu\neq 0$ for every $\nu\in A$, then $I_\alpha\subseteq \bigcap \ker f_\nu\neq 0$ since I_α is irreducible and E_α is an essential extension of I_α. Hence M_ν has an isomorphic image of E_α and M is a cogenerator.

Corollary 1. *A ring Λ is a self-cogenerator ring if and only if every E_α, i.e., $\sum^{\oplus} E_\alpha$ is projective.*

Proof. If $\sum^{\oplus} E_\alpha$ is projective, $\sum^{\oplus} E_\alpha < \oplus\sum^{\oplus}\Lambda$, and $\sum^{\oplus}\Lambda$ is a cogenerator. Hence Λ is a cogenerator by Lemma 1. Conversely if Λ is a cogenerator, $\Lambda\oplus > E_\alpha$, and E_α is projective.

Lemma 2. *If M is a cogenerator and $\mathfrak{l}$ is a left ideal of Λ, then $\mathfrak{l}=l_\Lambda(r_M(\mathfrak{l}))$. Hence every cogenerator is faithful. Conversely assume that Λ is a left self-cogenerator ring. Then every faithful module is a cogenerator.*