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Osaka Math. J.
11 (1959), 147-152.

Supplement to Note on Brauer's Theorem of Simple Groups. II

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The aim of this note is to complete the proof of the following theorem: *Let $\mathfrak{G}$ be a finite group which contains an element P of prime order p which commutes only with its own powers* (*condition* $(*)$) *and assume that $\mathfrak{G}$ is equal to its commutator-subgroup $\mathfrak{G}'$* (*condition* $(**)$). *Then the order g of $\mathfrak{G}$ is expressed as $g=p(p-1)(1+np)/t$, where $1+np$ is the number of conjugate subgroups of order p and t is the number of classes of conjugate elements of order p. If $n\leqq p+2$ and $t\not\equiv 0$ (mod. 2), then p is of the form $2^{\mu}-1$ and $\mathfrak{G}\cong LF(2, 2^{\mu})$.*

In [3], the theorem was proved for the case $n<p+2$ and $t\not\equiv 0$ (mod. 2): *If $n<p+2$ and $t\not\equiv 0$ (mod. 2), under $(*)$ and $(**)$, then p is of the form $2^{\mu}-1$ and $\mathfrak{G}\cong LF(2, 2^{\mu})$.* In [4], the case $n=p+2$ and $t\not\equiv 0$ (mod. 2) are discussed, but the equation in p. 230, line 6 is not correct[1], this value should be $\omega^{j(\mu+\nu)}\cdot(-1)^{jt}$. So the representation of degree $p+1$ may occur. Therefore, in this note, we shall assume that the irreducible representation of degree $p+1$ occurs besides the assumptions $(*)$, $(**)$, $n=p+2$ and $t\not\equiv 0$ (mod. 2). Under these assumptions we shall prove that such a group does not exist.

We shall use the same notations as Brauer [1]. First of all, we shall assume that $n=p+2=F(p, 1, 2)=F(p, u, 1)$ with positive integer u. For, if n does not have the expression $F(p, u, 1)$ with positive integer u, then the character-relations in $B_1(p)$ yields a contradiction easily. Simple computations show that the possible values of the irreducible characters in $B_1(p)$ are 1, $p+1$, $up+1$, $(u-1)p-1$, $(up+1)/t$ and $((u-1)p-1)/t$. In order to consider such characters, we shall prove following lemmas, essentially due to Brauer.

Lemma 1. *Under assumptions $(*)$, $(**)$, $n=p+2$ $t\not\equiv 0$ (mod. 2), if $\mathfrak{G}$ has an irreducible character A of degree $up+1 (u>1)$, then for the element I of order 2 in the normalizer $\mathfrak{N}(\mathfrak{P})$ of a p-Sylow subgroup $\mathfrak{P}$*

1) W. F. Reynolds kindly pointed out this error and gave the auther many useful suggestions. By this error, Theorem in [5] (p. 107) should be corrected as follows; If $2p-3<n\leqq 2p+3$, $t\not\equiv 0$ (mod. 2) and $t>1$, then $2p+1$ is a prime power and $\mathfrak{G}\cong LF(2, 2p+1)$, *unless the irreducible representation of degree $p+1$ occurs.* But Theorem in [5] (p. 116) is valid.