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Kernel Functions of Diffusion Equations II

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The present paper is a continuation of the author's previous paper "Kernel Functions of Diffusion Equation I", Osaka Mathematical Journal Vol. 9, 1957, pp. 201-214. Some notations which were defined in the previous paper will be used without repeating the definitions.

2. Suppose that D be regularly open and ∂D be smooth. Then Theorem 1 of the previous paper holds and $K(x, y; t)$ is a well defined continuous non-negative function, which is smaller than $E_t(x, y)$. In this paper the dimension d is assumed to be $\geqq 3$. Set

$$G(x, y) = \lim_{h\to 0}\int_h^\infty K(x, y; t)\,dt = \int_{+0}^\infty K(x, y; t)\,dt \tag{1}$$

Lemma 2.1. *$G(x, y)$ is the Green's function of the Laplacian over D with zero boundary.*

Proof. Take a C^2-function $\varphi(y)$ and set $\varphi_s(y) = \int_D K(x, y; s)\varphi(y)\,dy$ over D. Then

$$\begin{aligned}\Delta_x\int_D G(x, y)\varphi_s(y)\,dy &= \int_D \Delta_x G(x, y)\varphi_s(y)\,dy \\ &= \int_D\int_{+0}^\infty \Delta_x K(x, y; t+s)\,dt\,\varphi(y)\,dy \\ &= \int_D\int_{+0}^\infty \frac{\partial}{\partial t}K(x, y; t+s)\,dt\,\varphi(y)\,dy \\ &= \int_D \lim_{h\to 0} K(x, y; h+s)\varphi(y)\,dy \\ &= \lim\int_D K(x, y; h+s)\varphi(y)\,dy \\ &= \varphi_s(x)\,,\end{aligned} \tag{2}$$

Therefore by making s towards 0, we have the required relation, which proves the lemma.

Now take an arbitrary bounded open set D and consider an increasing sequence of bounded open sets $\{D_k\}$ with smooth boundaries converging to D. To each D_k we can associate the kernel function $K_k(x, y; t)$ which forms an increasing sequence of non-negative functions.