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Mass Distributions on the Ideal Boundaries of Abstract Riemann Surfaces, III

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In the previous paper we defined a function $N(z, p)$ and ideal boundary points and studied some properties of superharmonic functions in $\bar{R}$, but the mass distributions are only slightly discussed. In the present article, we rewrite pages from 174 to 176 of II[1] in more precise form and continue the previous work. We use the same notations and definitions as in II.

Theorem 1. *Let p be a minimal point and $v(p)$ be a neighbourhood of p. Let $V^M(z)$ be a harmonic function in $v(p)$ such that $V^M(z)=\min(M, N(z, p))$ on $\partial v(p)$ and $V^M(z)$ has M.D.I. over $v(p)$. Put $V(z)=\lim_{M=M'} V^M(z) : M'=\sup N(z,p)$. Then $N(z,p)-V(z)=N'(z,p)>0$ and $N'(z,p)$ has the same properties as $N(z,p)$.*

Suppose $\sup N(z,p)=\infty$, i.e. p is of capacity zero. Assume $V(z)\equiv N(z,p)$. Then $N(z,p)=\int_{\bar{R}-v(p)} N(z,q)\,d\mu(q)$. Since $N(z,p)$ is harmonic in R, $V(z)=\int_{B-v(p)} N(z,q)\,d\mu(q)$. If μ is a point mass, $N(z,p)=N(z,q) : q\notin v(p)$, which implies $p=q\notin v(p)$. This is a contradiction. Hence μ is not a point mass. Therefore there exist two positive mass distributions μ_1 and μ_2 such that $\mu=\mu_1+\mu_2$ and both $V_1(z)=\int N(z,q)d\mu_1(q)$ and $V_2(z)=\int N(z,q)d\mu_2(q)$ are not multiples of $N(z,p)$. Because, if every μ_i presents a multiple of $N(z,p)$ and whose kernel k_i tends to a point $q\notin v(p)$. Then $\lim_{i=\infty}\frac{\mu_i}{\text{total mass of }\mu_i}$ represents $N(z,p)=N(z,q) : q\notin v(p)$. This is also a contradiction. Therefore $N(z,p)-V_1(z)$ (>0) and $V_1(z)$ (>0) are superharmonic in $\bar{R}$, whence $N(z,p)$ is not minimal. Hence $V(z)<N(z,p)$. Next we show that $V(z)$ *has no mass at p* in any canonical mass distribution[2]. To the contrary, suppose $V(z)$ has a positive mass at p. Then

1) Z. Kuramochi: Mass distributions on the ideal boundaries, II. Osaka Math. Jour., 8, 1956.

2) At present we cannot prove the uniqueness of canonical mass distributions.