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*On the Invariants of Finite Nilpotent Groups**

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1. If $x_1, \cdots, x_n$ are independent variables over a field k operated on by a group G of linear transformations of finite order, and if $x_1^\sigma, \cdots, x_n^\sigma$ are the linear functions into which the variables are changed by a linear transformation of the group, then a rational function $F(x_1, \cdots, x_n)$ of variables is called an invariant of the group, if $F(x_1, \cdots, x_n) = F(x_1^\sigma, x_2^\sigma, \cdots, x_n^\sigma)$ for each linear transformation σ of the group. Under what condition is the field of invariants birational[1]? This is a very difficult question for general group G and field k.

In the present note, calculating the complete systems of generators of the invariants inductively, we shall prove the birationality of the fields of invariants for the particular type of representations of nilpotent groups. Our result is the following:

Theorem. *Let G be a finite nilpotent group of exponent N and let k be a field containing N-th roots of unity, where we assume that the characteristic of k is coprime to N. Let $G = G_0 \supset G_1 \supset \cdots \supset G_{r+1} = \{e\}$ be the descending chain of normal subgroups of G such that G_i/G_{i+1} is a cyclic subgroup of G/G_{i+1} $(i = 0, 1, \cdots, r)$. Let λ_i be the natural homomorphism of G onto G/G_i and let $\{M_i(\sigma)\}$ be the regular representation of G/G_i $(i = 1, 2, \cdots, r+1)$. Put*

$$M(\sigma) = \overbrace{M_1(\lambda_1(\sigma)) + \cdots + M_1(\lambda_1(\sigma))}^{t_1} \oplus \overbrace{M_2(\lambda_2(\sigma)) + \cdots + M_2(\lambda_2(\sigma))}^{t_2} \oplus$$
$$\cdots \oplus \overbrace{M_{r+1}(\sigma) + \cdots + M_{r+1}(\sigma)}^{t_{r+1}}, \quad \textit{where } t_i \geqq 1 \ (i = 1, 2, \cdots r+1).$$

Then the field of invariants of $\{M(\sigma)\,;\ \sigma \in G\}$ is birational.

2. The proof of the theorem.

We shall prove the theorem by the induction on r. For the case $r+1 = 1$ the theorem is evidently true. We assume that it is true for the case the length of the above descending chain of normal subgroups is less than $r+1$.

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*) The note was prepared while the author was a Yukawa Fellow at Osaka University.

1) We mean by birationality the purely transcendency over k.