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On the Pseudo-Harmonic Functions

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Introduction. Let F be an orientable surface. Let $u(p)$ be a real-valued function in a neighborhood N_{p_0} of p_0 on F where N_{p_0} corresponds to the unit circular disc in the complex plane by the topological mapping $z=T_{p_0}(p)$, $z=x+iy$.

Set
$$u(p)=u(T_{p_0}(p))=U(z)\,.$$

Then $u(p)$ is termed *pseudo-harmonic* at p_0, if $U(z)$ is harmonic and not identically constant in $|z|<1$. A real-valued function on F is termed *pseudo-harmonic* if it is pseudo-harmonic on each point of F. In this paper we will prove that there exist the local parameters such that F is a Riemann surface with respect to them and $u(p)$ is harmonic on F.

1. Terminologies and notations.

Let $u(p)$ be a pseudo-harmonic function on F. By the *level-curve* of $u(p)$ with the *height* c, we mean the locus of the equation $u(p)=c$. It is well known that with each point $p_0 \in F$, there exists a suitably chosen neighborhood N_{p_0} of p_0 and a topological mapping $z=T_{p_0}(p)$ of N_{p_0} onto $|z|<|1$ under which p_0 goes into $z=0$ and the level-curves of $u(p)$ in N_{p_0} go into the level-curves of $Re\, z^n$ in $|z|<1$[1]. we shall term this N_{p_0} a *canonical neighborhood* of p_0. When $n=1$, we shall call p_0 a *regular point* and N_{p_0} a *simple canonical neighborhood*. When $n \geqq 2$, we shall call p_0 a *saddle-point* of *order* n. A real-valued function $v(p)$ on F is called "*pseudo-conjugate to a pseudo-harmonic function* $u(p)$", if it satisfies the following condition.

There exists a topological mapping $z=T_{p_0}(p)$ by which N_{p_0} corresponds to $|z|<1$, and $U(z)=u(T_{p_0}(p))$ is conjugate-harmonic to $V(z)=v(T_{p_0}(p))$ in $|z|<1$.

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1) Y. Tôki, A topological characterization of pseudo-harmonic functions, Osaka Mathematical J. **3** (1951), 101–122. See also J. Jenkins and M. Morse, Topological methods on Riemann surface, pseudoharmonic function. Contributions to the theory of Riemann surfaces 1953 p. 114.