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Osaka J. Math.
41 (2004), 545–561

ANTI-SELF-DUAL HERMITIAN METRICS AND PAINLEVÉ III

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(Received January 20, 2003)

0. Introduction

The aim of this paper is to study the $SU(2)$-invariant anti-self-dual metrics which is specified by the solutions of Painlevé III. We study not only the diagonal metrics, but also the non-diagonal metrics.

Hitchin [6] shows that the $SU(2)$-invariant anti-self-dual metric is generically specified by a solution of Painlevé VI with two complex parameters.

Painlevé VI is shown to be a deformation equation for a linear problem

$$\left(\frac{d}{dz} - B_1\right)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0,$$

where B_1 has four simple poles on $\mathbb{CP}^1$ [7]. And Painlevé V, IV, III, II are degenerated from Painlevé VI:

$$\begin{tikzcd}
 & & \begin{matrix}\mathrm{IV}\\(3,1)\end{matrix} \arrow[dr] & \\
\begin{matrix}\mathrm{VI}\\(1,1,1,1)\end{matrix} \arrow[r] & \begin{matrix}\mathrm{V}\\(2,1,1)\end{matrix} \arrow[ur] \arrow[dr] & & \begin{matrix}\mathrm{II}\\(4)\end{matrix} \\
 & & \begin{matrix}\mathrm{III}\\(2,2)\end{matrix} \arrow[ur] &
\end{tikzcd}$$

This is the confluence diagram of poles of B_1, where the Roman numerals represent the types of the Painlevé equation, and the parenthesized numbers represent the orders of poles of B_1. For example, Painlevé III is shown to be a deformation equation for a linear problem with two double poles.

Hitchin used the twistor correspondence [1, 11] to associate the anti-self-dual equation and the Painlevé equation. On the twistor space, the lifted action of $SU(2)$ determines a pre-homogeneous action of $SU(2)$, and it determines an isomonodromic family of connections on $\mathbb{CP}^1$, and then we obtain the Painlevé equation.

Due to the reality condition of the twistor space, the poles of B_1 makes two antipodal pairs. Therefore, the configuration of poles becomes the type of Painlevé III or VI. Generically, the anti-self-dual metric is specified by a solution of Painlevé VI.

In this framework, Hitchin [6] classified the diagonal anti-self-dual metrics, and Dancer [5] shows that the diagonal scalar-flat Kähler metric is specified by a solution of Painlevé III with a parameter $(0, 4, 4, -4)$, where *diagonal* metric is in the shape of (1) in Section 1. Since the anti-self-dual Einstein metrics are diagonal, the classifi-