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FORMAL GEVREY THEORY FOR SINGULAR FIRST ORDER SEMI-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction and main result

In this paper we are concerned with formal power series solutions of the following first order semi-linear partial differential equation:

$$\begin{aligned} P(x, D)u(x) \equiv \sum_{i=1}^{d} a_i(x)D_iu(x) &= f(x, u(x)), \quad u(0) = 0, \\ x = (x_1, \dots, x_d) \in \mathbf{C}^d, &\quad D_i = \frac{\partial}{\partial x_i}, \end{aligned} \tag{1.1}$$

where coefficients $a_i(x)$ $(i = 1, \dots, d)$ and $f(x, u)$ are holomorphic in a neighborhood of $x = 0$ and $(x, u) = (0, 0)$, respectively.

If $a_i(0) \neq 0$ for some i, the solvability is well known by Cauchy-Kowalevsky's theorem. Therefore we shall study the case where

$$a_i(0) = 0 \quad \text{for all} \quad i = 1, \dots, d, \tag{1.2}$$

which is called a singular or degenerate case. In the following we always assume (1.2).

The first purpose of this paper is to prove the existence and the uniqueness of the formal power series solution $u(x) = \sum_{|\alpha|\geq 1} u_\alpha x^\alpha$ $(\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{N}^d$, $\mathbf{N} = \{0, 1, 2, \dots\}$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d})$ centered at the origin for the singular equation (1.1). As we will see later, we can prove it under some condition on the principal part $P(x, D)$. However, this formal power series solution $u(x)$ does not necessarily converge. So we would like to obtain the rate of divergence, which is called the Gevrey order, of the formal solution (cf. Definition 1.1). This is the second purpose of this paper.

1.1. Motivation. In the paper Hibino [2], we considered the following singular first order linear partial differential equation:

$$\widetilde{P}(x, D)u(x) \equiv \sum_{i=1}^{d} a_i(x)D_iu(x) + b(x)u(x) = f(x), \tag{1.3}$$