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GENERALIZED ALEXANDER DUALITY AND APPLICATIONS

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Introduction

Let Δ be a simplicial complex on the vertex set $[n]$, K a field, $S = K[x_1, \dots, x_n]$ the polynomial ring and $K[\Delta]$ the Stanley-Reisner ring over S. In a series of papers ([4], [5], [7]) relations between Betti numbers of $K[\Delta]$ and those of the Stanley-Reisner ring $K[\Delta^*]$ of the Alexander dual Δ^* have been studied.

In this paper we extend these results to squarefree S-modules, which were introduced by Yanagawa in [6]. This will be accomplished by defining the dual of a squarefree S-module. The definition is a natural extension of the Alexander dual.

To define the generalized Alexander dual we will see that there is an equivalence of the categories of squarefree S-modules and squarefree E-modules, where E denotes the exterior algebra. In the category of squarefree E-modules we may consider the E-dual $M^* = \mathrm{Hom}_E(M, E)$. If M is the squarefree E-module corresponding to a squarefree S-module N, then we call the squarefree S-module corresponding to M^* the generalized Alexander dual of N. The construction which assigns to a squarefree S-module a squarefree E-module is described in [1].

For the applications it is important to consider the so called distinguished pairs (l, j) introduced by Aramova and Herzog in [3]. Distinguished pairs are homological invariants of modules over the exterior algebra. The definition is based on the Cartan homology, an analogue to the Koszul homology in the polynomial ring. We generalize this definition to homological distinguished pairs (l, j) and cohomological distinguished pairs (l, j).

We prove that a homological distinguished pair (l, j) of M corresponds to the cohomological distinguished pair $(l, n-j)$ of M^*, which in turn corresponds to the homological distinguished pair $(l, n-j-l+1)$ of M^*. These homological considerations lead to the following results about graded Betti numbers:

Let $\beta_{i,i+j}$ be the graded Betti number of a finitely generated graded S-module. Bayer, Charalambous and S. Popescu introduced in [4] a refinement of the Mumford-Castelnuovo regularity, the extremal Betti numbers. They call a Betti number $\beta_{i,i+j} \neq 0$ extremal if $\beta_{l,l+r} = 0$ for all $r \geq j$ and all $l \geq i$ with $(l, r) \neq (i, j)$. One of their results states the following: if $\beta_{i,i+j}(K[\Delta])$ is an extremal Betti number of $K[\Delta]$, then $\beta_{j+1,i+j}(K[\Delta^*])$ is an extremal number of $K[\Delta^*]$ and $\beta_{i,i+j}(K[\Delta]) = \beta_{j+1,i+j}(K[\Delta^*])$.