ON HERSTEIN'S THEOREM CONCERNING THREE FIELDS

CARL FAITH[1]

Let $L > K \geqq \Phi$, $L \neq K$, be three fields such that: (1) L/K is not purely inseparable, and (2) L/Φ is transcendental. Then Herstein's theorem [2] asserts the existence of $u \in L$ such that $f(u) \notin K$ for every non-constant polynomial $f(X) \in \Phi[X]$. Thus Herstein's theorem can be given the following equivalent form:

THEOREM (Herstein). *If L, K, and Φ are three fields satisfying* (1) *and* (2), *$L \neq K$, then there exists $u \in L$ which is transcendental over Φ such that $K \cap \Phi[u] = \Phi$, where $\Phi[u]$ is the subring generated by Φ and u.*

The main part of Herstein's proof depends on a lemma of Nagata, Nakayama, and Tsuzuku in valuation theory of fields [*On an existence lemma in valuation theory*, Nagoya Math. Journal, vol. 6 (1953)]; the proof of this lemma in turn requires a knowledge of arithmetic in "algebraic number and function fields". In the present note I present an elementary proof of Herstein's theorem in which only the most basic properties of simple transcendental fields are used. In this development the result for the case $L = \Phi(x)$ is sharpened: then there exists a polynomial $q = q(x) \in \Phi[x]$ not in Φ such that $K \cap \Phi[q] = \Phi$.

Herstein's elementary reduction to the pure transcendental case constitutes a reduction for the theorem as stated above so we can assume that $L = \Phi(x)$. In this case it is known[2] that $K \cap \Phi[x]$ is finitely generated over Φ as a ring, for any intermediate field K. The proposition below gives a new proof and at the same time sharpens this result: *Then $K \cap \Phi[x]$ has a single generator over Φ.*

Received March 22, 1961.

[1] North Atlantic Treaty Organization (U.S.A.) Postdoctoral Fellow in the Mathematical Institute, Heidelberg University, on leave from Pennsylvania State University.

[2] This is the one dimensional solution to Hilbert's Fourteenth Problem. See [4] for Zariski's generalization and solution to the one and two dimensional cases of Hilbert's problem.