ON MONTEL'S THEOREM

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1. In this note we shall prove a theorem which is related to Montel's theorem [1] on bounded regular functions. Let E be a measurable set on the positive y-axis in the $z(=x+iy)$-plane, $E(a, b)$ be its part contained in $0 \leqq a \leqq y \leqq b$, and $|E(a, b)|$ be its measure. We define the lower density of E at $y=0$ by

$$\lambda = \varliminf_{r\to 0} \frac{|E(0, r)|}{r}.$$

LEMMA. *Let E be a set of positive lower density λ at $y=0$. Then E contains a subset E_1 of the same lower density at $y=0$ such that $E_1 \cup \{0\}$ is a closed set.*

Proof. Let $r_n = 1/n$ $(n = 1, 2, \dots)$. There exists a closed subset $E_1(r_{n+1}, r_n)$ of $E(r_{n+1}, r_n)$, such that

$$|E_1(r_{n+1}, r_n)| \geqq \delta_n |E(r_{n+1}, r_n)| \qquad (n=1, 2, \dots),$$

with $\delta_n = 1 - \dfrac{1}{n}$. We put

$$E_1 = \sum_{n=1}^{\infty} E_1(r_{n+1}, r_n).$$

Then if $r_n < r \leqq r_{n-1}$,

$$|E_1(0, r)| \geqq \sum_{i=n}^{\infty} |E_1(r_{i+1}, r_i)| \geqq \delta_n |E(0, r_n)|,$$

so that

$$\frac{|E_1(0, r)|}{r} \geqq \frac{|E_1(0, r_n)|}{r}\delta_n \geqq \frac{|E(0, r_n)|}{r_n} \cdot \frac{r_n}{r_{n-1}}\delta_n,$$

whence

$$\lambda = \varliminf_{r\to 0} \frac{|E(0, r)|}{r} \geqq \varliminf_{r\to 0} \frac{|E_1(0, r)|}{r} \geqq \varliminf_{n\to\infty} \frac{|E(0, r_n)|}{r_n} \geqq \lambda.$$

Hence

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