A THEOREM ON THE AFFINE TRANSFORMATION GROUP OF A RIEMANNIAN MANIFOLD

SHOSHICHI KOBAYASHI

1. Introduction

Every Riemannian manifold has a unique affine connection without torsion, which is necessarily invariant by any isometrical transformation of the manifold. However, an affine transformation (i.e., transformation leaving invariant the affine connection) is not necessarily an isometrical transformation. (Consider, for example, the ordinary Euclidean space).

Let us denote by M an n-dimensional Riemannian manifold and by $I(M)$ (resp. $A(M)$) the group of all isometrical (resp. affine) transformations of M. Then, in general, $I(M)$ is a closed subgroup of $A(M)$. Yano [2] proved that, if M is compact, then the connected component of the unit of $A(M)$ is contained in $I(M)$. Nomizu [1] obtained also some results concerning the relation between $A(M)$ and $I(M)$. The purpose of the present paper is to prove the following theorem: [1)]

THEOREM. *If M is an irreducible and complete Riemannian manifold, then $A(M)$ is equal to $I(M)$, except the case M is the* 1*-dimensional Euclidean space.*

2. Reduction of the problem

Let ds^2 be the Riemannian metric on M, which we are considering. If we denote by $\varphi^*(ds^2)$ the metric induced by an affine transformation φ, then we have the following lemma: [1]

LEMMA. *If M is irreducible, then $\varphi^*(ds^2) = c^2 \cdot ds^2$, where c is a positive constant.*

Therefore we have only to show that the constant c is equal to 1. The

Received March 21, 1955.

* This theorem has been proved independently by T. Nagano for the connected component of $A(M)$, using the theory of infinitesimal affine transformations and Killing vector fields (not yet published). J. Hano has also proved a similar result which generalizes the theorem of Yano (see his paper in this journal).