ON THE DIMENSION OF MODULES AND ALGEBRAS, II

(FROBENIUS ALGEBRAS AND QUASI-FROBENIUS RINGS)

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In this paper we study Frobenius algebras and quasi-Frobenius rings with particular emphasis on their cohomological dimensions. For definitions of these cohomological dimensions we refer the reader to Cartan-Eilenberg [3] or Eilenberg [4].

We consider (in §2) symmetric and Frobenius algebras in a setting more general than that of Brauer and Nesbitt [2], [13], and show (in §3) that for such algebras the cohomological dimension is either 0 or ∞.

These results are applied (§4) to the group ring $\Lambda = K(\Pi)$ where Π is a finite group of order r and K is a commutative ring. It is shown that Λ is symmetric and that $\dim \Lambda = 0$ if $rK = K$ and that $\dim \Lambda = \infty$ if $rK \neq K$.

The phenomenon that the cohomological dimension is either 0 or ∞ is again encountered (§5) in a ring Λ which is *left self-injective* i.e. a ring Λ which when regarded as a left Λ-module is injective. Such rings, under different terminologies have been considered recently in Ikeda [7], Nagao-Nakayama [10] and Ikeda-Nakayama [9] in connection with quasi-Frobenins algebras and rings. We further refine these results by showing (§§6, 7) that the notions "quasi-Frobenius ring" and "left self-injective ring" are equivalent for rings which are (left and right) Noetherian, or satisfy minimum condition for left or right ideals.

All rings considered have a unit element which operates as the identity on all modules considered.

§1. Duality

Let K be a commutative ring. For each K-module A we define the *dual* K-module

$$A^{\circ} = \mathrm{Hom}_K(A, K).$$

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