# NOTE ON SCHIFFER'S VARIATION IN THE CLASS OF UNIVALENT FUNCTIONS IN THE UNIT DISC

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**1.** Let $S$ denote the class of univalent functions $f(z)$ in the unit disc $D: |z| < 1$ with the following expansion:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots. \tag{1}$$

We denote by $f_n(z)$ the extremal function in $S$ which gives the maximum value of the real part of $a_n$ and by $D_n$ the image of $D$ under $w = f_n(z)$. Schiffer proved in his papers [1] and [2] by using his variational method that the boundary of $D_n$ consists of analytic slits $w = w(t)$, $t$ being a real parameter, satisfying

$$\left(\frac{dw}{dt}\right)^2 \frac{1}{w} \sum_{k=2}^{n} \frac{a_n^{(k)}}{w^k} < 0, \tag{2}$$

where $a_n^{(k)}$ is the $n$th coefficient of $f_n(z)^k = \sum_{\nu=k}^{\infty} a_\nu^{(k)} z^\nu$, so that follows from the Schwarz reflection principle

$$\frac{z^2 f_n'(z)^2}{f_n(z)} \sum_{k=2}^{n} \frac{a_n^{(k)}}{f_n(z)^k} = (n-1)a_n + \sum_{k=1}^{n-1} k\left(\frac{a_k}{z^{n-k}} + \bar{a}_k z^{n-k}\right) \tag{3}$$

in the $z$-plane. Thus the left-hand side of (3) is due to a variation of the range $D_n$. In this note, we shall show that the right-hand side of (3) is due to a variation of the domain $D$.

**2.** For a complex number $\gamma$, a real number $\tau$ and a sufficiently small $r > 0$, we consider the finite $w$-plane slit along the segment $S(\gamma; r, \tau)$ with end points $\gamma - re^{i\tau}$ and $\gamma + re^{i\tau}$ and denote it by $\Omega(\gamma; r, \tau)$. For $\omega$, $-1 < \omega < 1$, let $\Lambda^+(\gamma; r, \tau, \omega)$ and $\Lambda^-(\gamma; r, \tau, \omega)$ be the circular arcs with end points $\gamma - re^{i\tau}$ and $\gamma + re^{i\tau}$ where they make with $S(\gamma; r, \tau)$ inner angles being equal to $\pi\omega$. We denote by $\Delta(\gamma; r, \tau, \omega)$ the domain which is

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