K. Nomizu
Nagoya Math. J.
Vol. 60 (1976), 181-188

A CHARACTERIZATION OF THE VERONESE VARIETIES*

KATSUMI NOMIZU

Let $P^m(C)$ be the complex projective space of dimension m. In a previous paper [2] we have proved

THEOREM A. *Let f be a Kaehlerian immersion of a connected, complete Kaehler manifold M^n of dimension n into $P^m(C)$. If the image $f(\tau)$ of each geodesic τ in M^n lies in a complex projective line $P^1(C)$ of $P^m(C)$, then $f(M^n)$ is a complex projective subspace of $P^m(C)$, and f is totally geodesic.*

In the present note, we shall first provide a much simpler geometric proof of this result and then give a characterization of the Veronese varieties by means of the notion of circles in $P^m(C)$. Generally, a curve $x(t)$ with arc-length parameter t in a Riemannian manifold is called a circle if there exists a field of unit vectors Y_t along the curve, which, together with the unit tangent vectors X_t, satisfies the differential equations

$$\nabla_t X_t = kY_t \quad \text{and} \quad \nabla_t Y_t = -kX_t ,$$

where k is a positive constant (see [4]).

By the Veronese variety we mean the imbedding of $P^n(C)$ into $P^m(C)$, where $m = n(n+3)/2$, which is defined as follows. Let S^{2n+1} be the unit sphere in the complex vector space C^{n+1} with the standard hermitian inner product (z, w) and corresponding real inner product $\langle z, w\rangle = \mathrm{Re}\,(z, w)$. On the other hand, the set of all complex symmetric matrices of degree $n+1$ can be considered as the vector space C^{m+1}, where $m = n(n+3)/2$, in which the standard hermitian inner product can be expressed by

$$(A, B) = \mathrm{trace}\, A\bar{B} , \qquad A, B \in C^{m+1} .$$

The mapping v which takes $x \in C^{n+1}$ into $x\,{}^t x \in C^{m+1}$ maps S^{2n+1} into the

Received June 10, 1975.
* Work supported by NSF Grant GP-38582X.