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Nagoya Math. J.
Vol. 56 (1974), 13-17

ON THE DISTRIBUTION OF ZEROS OF A STRONGLY ANNULAR FUNCTION

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A function $f(z)$, regular in the unit disk D, is called annular ([1], p. 340) if there is a sequence of closed Jordan curves $J_n \subset D$ satisfying

(A_1) J_n is contained in the interior of J_{n+1} for every n,

(A_2) given $\varepsilon > 0$, there exists a positive number $n(\varepsilon)$ such that, for each $n > n(\varepsilon), J_n$ lies in the region $1 - \varepsilon < |z| < 1$ and

(A_3) $\lim_{n\to\infty} \min \{|f(z)|\,; z \in J_n\} = + \infty$.

One says that $f(z)$ is strongly annular if the J_n can be taken as circles concentric with the unit circle C. As for examples of annular functions, see ([4], p. 18).

Given a function $f(z)$ in D, denote by $Z(f)$ the set of zeros of $f(z)$ and $Z'(f)$ the set of limit points of $Z(f)$. If $f(z)$ is annular, $Z(f)$ is an infinite set of points of D ([1], p. 340) and clearly $Z'(f) \subset C$. In [1], Bagemihl and Erdös raised the following question: If $f(z)$ is annular, is $Z'(f) = C$? This question seems to be reasonable because many early examples of annular functions had this property. In [3], however, an example of an annular function $g(z)$ was constructed with $Z'(g) = \{1\}$. It is not known, regretfully, whether or not this example is strongly annular. Thus the problem of Bagemihl and Erdös remains open in the case where "annular" is replaced by "strongly annular" ([5], p. 141). In this note we shall give an example of a strongly annular function $f(z)$ with $Z'(f) = \{1\}$, modifying the technique for constructing the example of Barth and Schneider [3].

1. We shall first make some definitions. Given a, b and θ such that $0 < a < b < 1$ and $0 < \theta < \pi/2$, we consider the annular sector $D(a, b\,; \theta) = \{z \in D\,; a < |z| < b$ and $-\theta < \arg z < \theta\}$. Moreover, for c, θ_1 and θ_2 with $0 < c < 1$ and $-\pi/2 < \theta_2 < \theta_1 < \pi/2$, let $\sigma(c\,; \theta_2, \theta_1)$ denote the circular arc $\{z \in D\,; |z| = c$ and $\theta_2 \leqq \arg z \leqq \theta_1\}$. Now we are to state

Received March 22, 1974.