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N-MANIFOLDS CARRYING BOUNDED BUT NO DIRICHLET FINITE HARMONIC FUNCTIONS

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Among the most remarkable results in the theory of harmonic functions on Riemann surfaces is the strictness of the inclusion relations $O_G < O_{HB} < O_{HD}$, established by Ahlfors [1,2], Royden [2,4], and Tôki [8] two decades ago. Subsequently the strictness of the relations $O_G < O_{HP} < O_{HB}$ was shown and a somewhat simpler proof of $O_{HB} < O_{HD}$ given by Sario [5] and Tôki [9]. Here O_G is the class of parabolic surfaces, and O_{HP}, O_{HB}, O_{HD} stand for the classes of surfaces which do not carry nonconstant harmonic functions which are positive, bounded, or Dirichlet finite, respectively. The corresponding nonstrict inclusion relations extend readily to Riemannian manifolds of any dimension, and so does the strictness of $O_G < O_{HP} < O_{HB}$ (see e.g. Sario-Schiffer-Glasner [7] and Sario-Nakai [6]). In contrast, the strictness of $O_{HB} < O_{HD}$ has remained an open problem. The purpose of the present paper is to submit an example which solves the problem in the affirmative for an arbitrary dimension N. In the process we also obtain complete characterizations of the Poincaré N-ball in O_{HB} and O_{HD}. This manifold plays an important role in the harmonic and biharmonic classification theory.

1. For $N \geq 3$, consider the N-ball $B_\alpha^N = \{|x| < 1, x = (x^1, \cdots, x^N), ds\}$ with the Poincaré-type metric $ds = (1 - |x|^2)^\alpha |dx|$, α constant.

THEOREM 1. $B_\alpha^N \notin O_{HB} \Leftrightarrow \alpha < 1/(N-2)$, $N \geq 3$.

Proof. For the necessity, observe that a radial function $h(r)$, $r = |x|$, is harmonic if and only if

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